On Right Orthodox Γ-Semigroup

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Abstract

Let $S = \{a, b, c, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two nonempty sets. S is called a Γ-semigroup if $a\alpha b \in S$, for all $\alpha \in \Gamma$ and $a, b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$. An element $e \in S$ is said to be α-idempotent for some $\alpha \in \Gamma$ if $e\alpha e = e$. A Γ- semigroup S is called regular Γ-semigroup if each element of S is regular i.e, for each $a \in S$ there exists an element $x \in S$ and there exist $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. A regular Γ-semigroup S is called a right orthodox Γ-semigroup if for any α-idempotent e and β-idempotent f of S, $e\alpha f$ is a β-idempotent. In this paper we introduce ip - congruence on regular Γ-semigroup and ip - congruence pair on right orthodox Γ-semigroup and investigate some results relating this pair.

AMS Mathematics Subject Classification (2010): 20M17.

Keywords: Γ-Semigroup, right orthodox Γ-Semigroup, left partial congruence, ip - congruence, normal subsemigroup, ip - congruence pair.

1 Introduction

Let $S = \{a, b, c, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two nonempty sets. S is called a Γ-semigroup if

(i)$a\alpha b \in S$, for all $\alpha \in \Gamma$ and $a, b \in S$ and

(ii)$(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

A semigroup can be considered to be a Γ-semigroup in the following sense. Let S be an arbitrary semigroup. Let 1 be a symbol not representing any element of S. Let us extend the binary operation defined on S to $S \cup \{1\}$ by defining $11 = 1$ and $1a = a1$ for all $a \in S$. It can be shown that $S \cup \{1\}$ is a semigroup with identity element 1. Let $\Gamma = \{1\}$. If we take $ab = a1b$, it can be shown that the semigroup S is a $\Gamma-$semigroup where $\Gamma = \{1\}$.

In [1] we introduced right orthodox Γ-semigroup. In [2] Gomes introduced the notion of congruence pair on orthodox semigroup and studied some of its properties. In this paper we introduce the notion of ip - congruence on regular Γ-semigroup, ip - congruence pair on right orthodox Γ-semigroup and studied some of its properties. We now recall some definition and results.

Definition 1.1 Let S be a Γ-semigroup. An element $a \in S$ is said to be regular if $a \in a\Gamma S\Gamma a$ where $a\Gamma S\Gamma a = \{a\alpha b\beta a : b \in S, \alpha, \beta \in \Gamma\}$. S is said to be regular if every element of S is regular.

Example 1.2 *[6] Let M be the set of all 3×2 matrices and Γ be the set of all 2×3 matrices over a field. Then M is a regular Γ semigroup.*

Example 1.3 *Let S be a set of all negative rational numbers. Obviously S is not a semigroup under usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p} : p \text{ is prime }\}$. Let $a, b, c \in S$ and $\alpha \in \Gamma$. Now if $a\alpha b$ is equal to the usual product of rational numbers a, α, b, then $a\alpha b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence S is a Γ-semigroup. Let $a = \frac{m}{n} \in S$ where $m > 0$ and $n < 0$. Suppose $m = p_1 p_2 \ldots\ldots\ldots p_k$ where p_i's are prime. Now $\frac{p_1 p_2 \cdots\cdots\cdots p_k}{n} (-\frac{1}{p_1}) \frac{n}{p_2 \cdots\cdots\cdots p_{k-1}} (-\frac{1}{p_k}) \frac{m}{n} = \frac{p_1 p_2 \cdots\cdots\cdots p_k}{n}$. Thus taking $b = \frac{n}{p_2 \cdots\cdots\cdots p_{k-1}}$, $\alpha = (-\frac{1}{p_1})$ and $\beta = (-\frac{1}{p_k})$ we can say that a is regular. Hence S is a regular Γ-semigroup.*

Definition 1.4 Let S be a Γ-semigroup and $\alpha \in \Gamma$. Then $e \in S$ is said to be an α-idempotent if $e\alpha e = e$. The set of all α-idempotents is denoted by E_α. We denote $\bigcup\limits_{\alpha \in \Gamma} E_\alpha$ by $E(S)$. The elements of $E(S)$ are called idempotent element of S.

Definition 1.5 Let S be a Γ-semigroup and $a, b \in S$, $\alpha, \beta \in \Gamma$. b is said to be an (α, β)-inverse of a if $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$. This is denoted by $b \in V_\alpha^\beta(a)$.

Definition 1.6 Let S be a Γ-semigroup. An equivalence relation ρ on S is said to be a *right (left) congruence* on S if $(a, b) \in \rho$ implies $(a\alpha c, b\alpha c) \in \rho$, $((c\alpha a, c\alpha b) \in \rho)$ for all $a, b, c \in S$

ISSN: 2349-7955
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and for all $\alpha \in \Gamma$. An equivalence relation which is both left and right congruence on S is called *congruence* on S.

Theorem 1.7 Let S be a regular Γ-semigroup and $a \in S$. Then $V_\alpha^\beta(a)$ is non-empty for some $\alpha, \beta \in \Gamma$.

Proof: Since S is regular there exist $b \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha b\beta a$. We consider the element $b\beta a\alpha b$. Now $a\alpha(b\beta a\alpha b)\beta a = (a\alpha b\beta a)\alpha b\beta a = a\alpha b\beta a = a$ and $(b\beta a\alpha b)\beta a\alpha(b\beta a\alpha b) = b\beta(a\alpha b)\beta a)\alpha b\beta a\alpha b = b\beta a\alpha b\beta a\alpha b = b\beta a\alpha b$. Hence $b\beta a\alpha b \in V_\alpha^\beta(a)$.

Definition 1.8 *[1]* A regular Γ-semigroup S is called a right orthodox Γ-semigroup if for any α-idempotent e and β-idempotent f of S, $e\alpha f$ is a β-idempotent.

Example 1.9 *[1] Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. S denotes the set of all mappings from A to B. Here members of S will be described by the images of the elements 1, 2, 3. For example the map $1 \to 4$, $2 \to 5$, $3 \to 4$ will be written as $(4, 5, 4)$ and $(5, 5, 4)$ denotes the map $1 \to 5$, $2 \to 5$, $3 \to 4$. A map from B to A will be described in the same fashion. For example $(1, 2)$ denotes $4 \to 1$, $5 \to 2$. Now $S = \Big\{(4, 4, 4), (4, 4, 5), (4, 5, 4), (4, 5, 5), (5, 5, 5), (5, 4, 5), (5, 4, 4), (5, 5, 4)\Big\}$ and let $\Gamma = \{(1, 1), (1, 2), (2, 3), (3, 1)\}$. Let $f, g \in S$ and $\alpha \in \Gamma$. We define $f\alpha g$ by $(f\alpha g)(a) = f\alpha\big(g(a)\big)$ for all $a \in A$. So $f\alpha g$ is a mapping from A to B and hence $f\alpha g \in S$ and we can show that $(f\alpha g)\beta h = f\alpha(g\beta h)$ for all $f, g, h \in S$ and $\alpha, \beta \in \Gamma$. We can show that each element x of S is an α-idempotent for an $\alpha \in \Gamma$ and hence each element is regular. Thus S is a regular Γ-semigroup. Moreover we can show that it is a right orthodox Γ-semigroup.*

Definition 1.10 *[1]* A regular Γ-semigroup M is a right orthodox Γ-semigroup if and only if for $a, b \in S$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma$, $a' \in V_{\alpha_1}^{\alpha_2}(a)$ and $b' \in V_{\beta_1}^{\beta_2}(b)$, we have $b'\beta_2 a' \in V_{\beta_1}^{\alpha_2}(a\alpha_1 b)$.

Theorem 1.11 Let S be a regular Γ-semigroup and E_α be the set of all α-idempotents in S. Let $e \in E_\alpha$ and $f \in E_\beta$. Then

$$RS(e, f) = \Big\{g \in V_\beta^\alpha(e\alpha f) \cap E_\alpha : g\alpha e = f\beta g = g\Big\}$$

is non-empty.

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Proof: Since S is regular, there exist $b \in S$ and $\gamma, \delta \in \Gamma$ such that $e\alpha f\gamma b\delta e\alpha f = e\alpha f$ and $b\delta e\alpha f\gamma b = b$. Now $(e\alpha f)\beta(f\gamma b\delta e)\alpha(e\alpha f) = e\alpha f\gamma b\delta e\alpha f = e\alpha f$ and $(f\gamma b\delta e)\alpha(e\alpha f)\beta(f\gamma b\delta e) = f\gamma b\delta e\alpha f\gamma b\delta e = f\gamma b\delta e$. Hence $f\gamma b\delta e \in V_\beta^\alpha(e\alpha f)$. Thus $V_\beta^\alpha(e\alpha f) \neq \phi$. Now let $x \in V_\beta^\alpha(e\alpha f)$ and setting $g = f\beta x\alpha e$ we have $g\alpha g = (f\beta x\alpha e)\alpha(f\beta x\alpha e) = f\beta(x\alpha e)\alpha f\beta x)\alpha e = f\beta x\alpha e = g$. Thus $g \in E_\alpha$.

Again $g\alpha e\alpha f\beta g = f\beta x\alpha e\alpha e\alpha f\beta f\beta x\alpha e = f\beta x\alpha e\alpha f\beta x\alpha e = f\beta x\alpha e = g$ and $e\alpha f\beta g\alpha e\alpha f = e\alpha f\beta f\beta x\alpha e\alpha e\alpha f = e\alpha f\beta x\alpha e\alpha f = e\alpha f$ implies that $g \in V_\beta^\alpha(e\alpha f)$. Hence $g\alpha e = f\beta x\alpha e\alpha e = f\beta x\alpha e = g$ and $f\beta g = f\beta f\beta x\alpha e = f\beta x\alpha e = g$. Therefore $RS(e, f) \neq \emptyset$.

Definition 1.12 Let S be a regular Γ- semigroup and e and f be α and β-idempotents respectively. Then the set $RS(e, f)$ described in the above theorem is called the right sandwich set of e and f.

Theorem 1.13 Let S be a regular Γ-semigroup and e and f be α and β-idempotents respectively. Then the set $RS(e, f) = \{g \in V_\beta^\alpha(e\alpha f) : g\alpha e = g = f\beta g \text{ and } e\alpha g\alpha f = e\alpha f\}$.

Proof: Let $P = \{g \in V_\beta^\alpha(e\alpha f) : g\alpha e = g = f\beta g \text{ and } e\alpha g\alpha f = e\alpha f\}$ and let $g \in RS(e, f)$. Then $g \in E_\alpha, g\alpha e = g = f\beta g$ and $g \in V_\beta^\alpha(e\alpha f)$. Now $e\alpha g\alpha f = e\alpha g\alpha e\alpha f\beta g\alpha f = e\alpha f\beta g\alpha e\alpha f\beta g\alpha e\alpha f = e\alpha f\beta g\alpha e\alpha f = e\alpha f$. Hence $RS(e, f) \subseteq P$. Next let $g \in P$. Now $g\alpha g = g\alpha e\alpha f\beta g = g$. Hence $g \in E_\alpha$, which shows that $P \subseteq RS(e, f)$ and hence the proof.

Theorem 1.14 Let S be a regular Γ- semigroup and $a, b \in S$.If $a' \in V_\alpha^\beta(a), b' \in V_\gamma^\delta(b)$ and $g \in RS(a'\beta a, b\gamma b')$ then $b'\delta g\alpha a' \in V_\gamma^\beta(a\alpha b)$.

Proof: Let $e = a'\beta a$ and $f = b\gamma b'$. Then e be an α-idempotent and$f$ is a δ-idempotent and also g is an α-idempotent. Now $(a\alpha b)\gamma(b'\delta g\alpha a')\beta(a\alpha b) = a\alpha f\delta g\alpha e\alpha b = a\alpha g\alpha b = a\alpha a'\beta a\alpha g\alpha b\gamma b'\delta b = a\alpha e\alpha g\alpha e\alpha b = a\alpha e\alpha f\delta b = a\alpha a'\beta a\alpha b\gamma b'\delta b = a\alpha b$. Again $(b'\delta g\alpha a')\beta(a\alpha b)\gamma(b'\delta g\alpha a') = b'\delta g\alpha e\alpha f\delta g\alpha a' = b'\delta g\alpha g\alpha a' = b'\delta g\alpha a'$. Hence $b'\delta g\alpha a' \in V_\gamma^\beta(a\alpha b)$.

Corollary 1.15 For $a, b \in S$, if $V_\alpha^\beta(a)$ and $V_\gamma^\delta(b)$ are nonempty then $V_\gamma^\beta(a\alpha b)$ is nonempty.

Proof: Let $a' \in V_\alpha^\beta(a)$ and $b' \in V_\gamma^\delta(b)$ then we know that $RS(a'\beta a, b\gamma b') \neq \phi$. For $g \in RS(a'\beta a, b\gamma b')$ and hence we get $b'\delta g\alpha a' \in V_\gamma^\beta(a\alpha b)$. Hence the proof.

Throughout our discussion we consider S as a regular Γ-semigroup.

2 Ip - congruence pair on right orthodox Γ-semigroup

In this section we characterize some congruence on a right orthodox Γ-semigroup.

Definition 2.1 Let S be a Γ-semigroup. A nonempty subset K of S is said to be partial Γ-subsemigroup if for $a, b \in K, a\alpha b \in K$, whenever $V_\alpha^\beta(a) \neq \phi$. for some $\alpha, \beta \in \Gamma$.

Definition 2.2 A partial Γ-subsemigroup K of S is said to be regular if $V_\alpha^\beta(k) \subseteq K$ for all $k \in K$ and $\alpha, \beta \in \Gamma$.

Definition 2.3 A partial Γ-subsemigroup K is said to be full if $E(S) \subseteq K$ where $E(S)$ is the set of all idempotent elements of S.

Definition 2.4 A partial Γ-subsemigroup K of S is said to be self conjugate if for all $a \in S, k \in K$ and $a' \in V_\alpha^\beta(a), a'\beta k\gamma a \in K$ whenever $V_\gamma^\delta(k) \neq \phi$ for some $\gamma, \delta \in \Gamma$.

Definition 2.5 A partial Γ-subsemigroup K of S is said to be normal if it is regular, full and self conjugate.

Definition 2.6 An equivalence relation ρ on S is said to be left partial congruence if $(a, b) \in \rho$ implies $(c\alpha_3 a, c\alpha_3 b) \in \rho$ whenever $V_{\alpha_3}^{\beta_3}(c)$ is nonempty. Note that every left congruence is a left partial congruence.

Here we consider these left partial congruence which satisfy the following condition: $(a, b) \in \rho$ implies $(a\alpha_1 c, b\alpha_2 c) \in \rho$ whenever each of the sets $V_{\alpha_1}^{\beta_1}(a), V_{\alpha_2}^{\beta_2}(b)$ is nonempty for $\alpha_i, \beta_i \in \Gamma, i = 1, 2$. We call this left partial congruence as inverse related partial congruence (ip - congruence).

Example 2.7 Let us consider the example given in Example 1.9. We now give a partition $S = \bigcup_{1 \le i \le 5} S_i$ and let ρ be the equivalence relation yielded by the partition where each S_i is given by:

$S_1 = \{(4, 4, 4)\}$,

$S_2 = \{(5, 5, 5)\}$,

$S_3 = \{(4, 5, 4), (5, 4, 5)\}$,

$S_4 = \{(4, 5, 5), (5, 4, 4)\}$,

$S_5 = \{(4, 4, 5), (5, 5, 4)\}$.

Here we see that $(4,5,4)\rho(5,4,5)$ but $(4,5,4)(3,1)(4,4,4) = (4,4,4)$ and $(5,4,5)(3,1)(4,4,4) = (5,5,5)$ i.e ρ is not a congruence.

Now for $f \in S$ we observe the following cases:

(a) $(4,4,4)\alpha f = (4,4,4)$ for all $\alpha \in \Gamma$,

(b) $(5,5,5)\alpha f = (5,5,5)$ for all $\alpha \in \Gamma$,

(c) $(4,5,4)(1,2)f = f$ and $(4,5,4)(2,3)f = f'$,

$(5,4,5)(2,3)f = f$ and $(5,4,5)(1,2)f = f'$,

(d) $(4,4,5)(2,3)f = f$ and $(4,4,5)(3,1)f = f'$,

$(5,5,4)(3,1)f = f$ and $(5,5,4)(2,3)f = f'$,

(e) $(4,5,5)(1,2)f = f$ and $(4,5,5)(3,1)f = f'$,

$(5,4,4)(3,1)f = f$ and $(5,4,4)(1,2)f = f'$,

From the above cases we can easily verify that ρ is an ip - congruence on S.

Definition 2.8 An ip - congruence ξ on $E(S)$ of S is said to be normal if for any α-idempotent e and β-idempotent $f, a \in S$ and $a' \in V_{\gamma}^{\delta}(a)$, $(e,f) \in \xi$ implies $(a'\delta e\alpha a, a'\delta f\beta a) \in \xi$ whenever $a'\delta e\alpha a, a'\delta f\beta a \in E(S)$.

Let ρ be an ip - congruence on a regular Γ - semigroup S. Then we can define a binary operation on S/ρ as $(a\rho)(b\rho) = (a\alpha b)\rho$ whenever $V_{\alpha}^{\beta}(a) \neq \phi$ for some $\beta \in \Gamma$. Note that $V_{\alpha}^{\beta}(a) \neq \phi$ for some $\alpha, \beta \in \Gamma$, because S is a regular Γ-semigroup. The operation is well defined because if $a\rho = a'\rho$ and $b\rho = b'\rho$ then

$$\begin{aligned}
(a\rho)(b\rho) &= (a\alpha b)\rho \text{ (Since } V_{\alpha}^{\beta}(a) \neq \phi \text{ for some } \alpha, \beta \in \Gamma) \\
&= (a\alpha b')\rho \\
&= (a'\alpha_1 b')\rho (\text{Since } V_{\alpha_1}^{\beta_1}(a') \neq \phi \text{ for some } \alpha_1, \beta_1 \in \Gamma) \\
&= (a'\rho)(b'\rho).
\end{aligned}$$

Using Corollary 1.15 we can say that the operation is easily seen to be associative, and so S/ρ is a semigroup.

Definition 2.9 Let ρ be an ip - congruence on a regular Γ-semigroup S. Let $\alpha \in \Gamma$, then the subset $\{a \in S : a\rho \in E(S/\rho)\}$ of S is called the kernel of ρ and it is denoted by $Ker\rho$.

Definition 2.10 Let ρ be an ip - congruence on a regular Γ-semigroup S. Then the restriction of ρ to the subset $E(S)$ is called the trace of ρ and it is denoted by $tr\rho$.

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Theorem 2.11 Let ρ be an ip - congruence on a regular Γ-semigroup S. Let $a, b \in S$ and suppose that $a' \in V_\alpha^\beta(a), b' \in V_\gamma^\delta(b)$ are such that $a'\beta b \in Ker\rho$ and either $(a\alpha a', b\gamma b'\delta a\alpha a') \in \rho$ or $(b\gamma b', a\alpha a'\beta b\gamma b') \in \rho$. Then $b\gamma a' \in Ker\rho$.

Proof: Let $a'\beta b \in Ker\rho$ for some $a' \in V_\alpha^\beta(a)$ and let $a'\beta b\rho e$ for some μ- idempotent e. If $(a\alpha a', b\gamma b'\delta a\alpha a') \in \rho$ for some $b' \in V_\gamma^\delta(b)$ then $(b\gamma a', b\gamma a'\beta b\gamma b'\delta a\alpha a') \in \rho$ which implies $(b\gamma a', b\gamma a'\beta b\gamma b'\delta a\alpha a') \in \rho$ and so

$$\begin{aligned}
(b\gamma a')\beta(b\gamma a') \quad & \rho \quad ((b\gamma a')\beta b\gamma b'\delta a')\beta(b\gamma a'\beta b\gamma b'\delta a\alpha a') \\
& \rho \quad (b\gamma a')\beta(b\gamma b')\delta a\alpha(a'\beta b)\gamma(a'\beta b)\gamma b'\delta a\alpha a' \\
& \rho \quad (b\gamma a')\beta b\gamma b'\delta a\alpha e\mu e\mu b'\delta a\alpha a' \\
& \rho \quad b\gamma a'\beta b\gamma b'\delta a\alpha e\mu b'\delta a\alpha a' \\
& \rho \quad b\gamma a'\beta b\gamma b'\delta a\alpha a'\beta b\gamma b'\delta a\alpha a' \\
& = \quad (b\gamma a'\beta b\gamma b'\delta a\alpha a')\beta(b\gamma b'\delta a\alpha a') \\
& \rho \quad (b\gamma a')\beta(a\alpha a') \\
& ,= \quad b\gamma a'.
\end{aligned}$$

Hence $b\gamma a' \in Ker\rho$.

If we now suppose that $(b\gamma b', a\alpha a'\beta b\gamma b') \in \rho$ for some $b' \in V_\gamma^\delta(b)$ then $(b\gamma a', a\alpha a'\beta b\gamma a') \in \rho$ and so

$$\begin{aligned}
(b\gamma a')\beta(b\gamma a') \quad & \rho \quad (a\alpha a')\beta b\gamma a')\beta(a\alpha a'\beta b\gamma a') \\
& = \quad a\alpha(a'\beta b\gamma a'\beta b)\gamma a' \\
& \rho \quad a\alpha e\mu e\mu a' \\
& = \quad a\alpha e\mu a' \\
& \rho \quad a\alpha a'\beta b\gamma a' \\
& \rho \quad b\gamma a'.
\end{aligned}$$

Hence $b\gamma a' \in Ker\rho$.

Theorem 2.12 If ρ be an ip - congruence on a regular Γ-semigroup S then for all $a, b \in S$ if there exist $a' \in V_\alpha^\beta(a)$ and $b' \in V_\gamma^\delta(b)$ such that $a'\beta b \in Ker\rho$, $(a\alpha a', b\gamma b'\delta a\alpha a') \in \rho$ and $(b'\delta b, b'\delta b\gamma a'\beta a) \in \rho$ then $(a, b) \in \rho$.

Proof: Let us suppose that $a, b \in S$ are such that for some $a' \in V_\alpha^\beta(a)$ and $b' \in V_\gamma^\delta(b)$ such that $a'\beta b \in Ker\rho$, $(a\alpha a', b\gamma b'\delta a\alpha a') \in \rho$ and $(b'\delta b, b'\delta b\gamma a'\beta a) \in \rho$. Then

$$\begin{aligned} a &= a\alpha a'\beta a \\ &\rho\ b\gamma b'\delta a\alpha a'\beta a \\ &= b\gamma b'\delta a. \end{aligned}$$

and

$$\begin{aligned} b &= b\gamma b'\delta b \\ &\rho\ b\gamma b'\delta b\gamma a'\beta a \\ &= b\gamma a'\beta a. \end{aligned}$$

Now by Theorem 2.11 $(b\gamma a')\rho \in E_\beta(S/\rho)$. Let $t\rho \in RS((b\gamma a')\rho, (a\alpha a')\rho)$. Now let $x\rho = (a'\rho)\beta(t\rho)\beta((b\gamma a')\rho)$, then

$$\begin{aligned} x\delta b\gamma x &\ \rho\ a'\beta t\beta b\gamma a'\beta b\gamma a'\beta t\beta b\gamma a' \\ &\ \rho\ a'\beta t\beta t\beta b\gamma a' \\ &\ \rho\ a'\beta t \\ &\ \rho\ a'\beta t\beta b\gamma a' \\ &\ \rho\ x. \end{aligned}$$

and

$$\begin{aligned} b\gamma x\beta b &\ \rho\ b\gamma a'\beta t\beta b\gamma a'\beta b \\ &\ \rho\ b\gamma a'\beta t\beta b \\ &\ \rho\ b\gamma a'\beta t\beta b\gamma a'\beta a \\ &\ \rho\ b\gamma a'\beta t\beta a \\ &\ \rho\ b\gamma a'\beta t\beta a\alpha a'\beta a \\ &\ \rho\ b\gamma a'\beta a\alpha a'\beta a \\ &\ \rho\ b\gamma a'\beta a \\ &\ \rho\ b. \end{aligned}$$

Hence $x\rho \in V_\gamma^\beta(b\rho)$. Now

$$\begin{aligned} (b\rho)\gamma(x\rho)\beta(a\rho) &= (b\gamma a')\rho\beta t\rho\beta(b\gamma a')\rho\beta(a\alpha a')\rho\beta(a\rho) \\ &= (b\gamma a')\rho\beta(a\alpha a')\rho\beta(a\rho) \\ &= (b\gamma a')\rho\beta(a\rho) \\ &= b\rho. \end{aligned}$$

Also $(b\rho)\gamma(b'\rho) = (b\rho)\gamma(x\rho)\beta(b\rho)\gamma(b'\rho)$. Hence

$$\begin{aligned} a\rho &= (b\rho)\gamma(b'\rho)\delta(a\rho) \\ &= (b\rho)\gamma(x\rho)\beta(b\rho)\gamma(b'\rho)\delta(a\rho) \\ &= (b\rho)\gamma(x\rho)\beta(a\rho) \\ &= b\rho. \end{aligned}$$

Thus $(a,b) \in \rho$.

We now treat S as a right orthodox Γ-semigroup throughout the paper.

Definition 2.13 A pair (ξ, K) consisting of a normal ip - congruence ξ on $E(S)$ and a normal partial Γ- subsemigroup K of S is said to be ip - congruence pair for S if for all $a, b \in S, a' \in V_{\alpha}^{\beta}(a)$ and $e \in E_{\gamma}$

(A) $e\gamma a \in K, (e, a\alpha a') \in \xi \Rightarrow a \in K$

(B) $a \in K \Rightarrow (a'\beta e\gamma a, a'\beta a'\beta e\gamma a\alpha a) \in \xi$

Theorem 2.14 If (ξ, k) is an ip - congruence pair for S then for $a' \in V_{\alpha}^{\beta}(a)$,$b' \in V_{\gamma}^{\delta}(b)$ and $e \in E_{\mu_1}$ and $f \in E_{\mu_2}$

(i) $a\alpha b \in K,\ (a'\beta a, b\gamma b'\delta a'\beta a) \in \xi \Rightarrow a\alpha e\mu_1 b \in K$

(ii) $a \in K,\ (a\alpha a', f) \in \xi \Rightarrow (f\mu_2 e\mu_1 f, f\mu_2 a'\beta e\mu_1 a\alpha f) \in \xi$ whenever $f\mu_2 a'\beta e\mu_1 a\alpha f \in E(S)$

Proof: Let $a\alpha b \in K$ and $(a'\beta a, b\gamma b'\delta a'\beta a) \in \xi$. Then

$$\begin{aligned} (b'\delta e)\mu_1(e\mu_1 b)\gamma(b'\delta e) &= b'\delta(b\gamma b'\delta e)\mu_1(b\gamma b'\delta e) \\ &= b'\delta(b\gamma b'\delta e) \text{ (Since } S \text{ is right orthodox)} \\ &= b'\delta e. \end{aligned}$$

and

$$\begin{aligned} (e\mu_1 b)\gamma(b'\delta e)\mu_1(e\mu_1 b) & \quad (e\mu_1 b\gamma b')\delta e\mu_1 b\gamma b')\delta b \\ & \quad (e\mu_1 b\gamma b')\delta b \\ & \quad e\mu_1 b. \end{aligned}$$

Hence $b'\delta e \in V_{\gamma}^{\mu_1}(e\mu_1 b)$. Similarly $e\mu_1 a' \in V_{\mu_1}^{\beta}(a\alpha e)$. Again

$$\begin{aligned} & (b'\delta e\mu_1 a')\beta(a\alpha e\mu_1 a')\beta(a\alpha e\mu_1 b)\gamma(b'\delta e\mu_1 a') \\ = \ & (b'\delta e)\mu_1(e\mu_1 b)\gamma(b'\delta e)\mu_1 a'\beta a\alpha e\mu_1 b\gamma b'\delta(e\mu_1 a')\beta(a\alpha e)\mu_1(e\mu_1 a') \\ = \ & b'\delta e\mu_1(e\mu_1 b\gamma b'\delta e\mu_1 a'\beta a)\alpha(e\mu_1 b\gamma b'\delta e a'\beta a)\alpha e\mu_1 e\mu_1 a' \\ = \ & b'\delta e\mu_1 e\mu_1 b\gamma b'\delta e\mu_1 a'\beta a\alpha e\mu_1 e\mu_1 a' \\ = \ & b'\delta e\mu_1 a'\beta a\alpha e\mu_1 e\mu_1 a' \\ = \ & b'\delta e\mu_1 a' \end{aligned}$$

Similarly we have $(a\alpha e\mu_1 b)\gamma(b'\delta e\mu_1 a')\beta(a\alpha e\mu_1 b) = a\alpha e\mu_1 b$ and we have $b'\delta e\mu_1 a' \in V_\gamma^\beta(a\alpha e\mu_1 b)$. Now $(a\alpha e\mu_1 b)\gamma(b'\delta e\mu_1 a') = a\alpha(a'\beta a)\alpha e\mu_1 b\gamma b'\delta e)\mu_1 a' \,\xi\, a\alpha(b\gamma b'\delta a'\beta a)\alpha(e\mu_1 b\gamma b'\delta e)\mu_1 a'$ since ξ is normal. Moreover $(a\alpha b)\gamma(b'\delta a')\beta(a\alpha e\mu_1 b)\gamma(b'\delta e\mu_1 a') \in E_\beta$ since $b'\delta a' \in V_\gamma^\beta(a\alpha b)$ and $(a\alpha b\gamma b'\delta a'\beta a\alpha e\mu_1 b\gamma b'\delta e\mu_1 a')\beta(a\alpha e\mu_1 b) = (a\alpha b)\gamma(b'\delta(a'\beta a\alpha e\mu_1 b\gamma b'\delta e\mu_1 a'\beta a\alpha e)\mu_1 b) \in K$ since $a\alpha b \in K$. Hence $a\alpha e\mu_1 b \in K$ by (A).

Let $a \in K$ and $(a\alpha a', f) \in \xi$. Then by condition (B) we have $(a'\beta e\mu_1 a, a'\beta a'\beta e\mu_1 a\alpha a) \in \xi$. Therefore

$$\begin{aligned} f\mu_2 e\mu_1 f &\;\xi\; a\alpha a'\beta e\mu_1 f \\ &\;\xi\; a\alpha a'\beta e\mu_1 a\alpha a' \\ &= (a'\beta e\mu_1 a)\alpha a' \\ &\;\xi\; a\alpha(a'\beta a'\beta e\mu_1 a\alpha a)\alpha a' (\text{Since } \xi \text{ is normal }) \\ &\;\xi\; f\mu_2 a'\beta e\mu_1 a\alpha f. \end{aligned}$$

Hence the proof.

Given such a pair (ξ, K) we define a binary relation $\rho_{(\xi,K)}$ on S by $(a,b) \in \rho_{(\xi,K)}$ if and only if there exist $a' \in V_\alpha^\beta(a)$, $b' \in V_\gamma^\delta(b)$, $a'\beta b \in K, (a\alpha a', b\gamma b'\delta a\alpha a') \in \xi, (b'\delta b, b'\delta b\gamma a'\beta a) \in \xi$.

To simplify the notation, given a congruence pair (ξ, K) we denote $\rho_{(\xi,K)}$ simply by ρ unless otherwise stated.

Theorem 2.15 Let (ξ, K) be an ip - congruence pair for S and $a, b \in S$ be such that $(a,b) \in \rho$ if and only if for all $a^* \in V_\alpha^\beta(a)$, $b^* \in V_\gamma^\delta(b)$, $a^*\beta b \in K, (a\alpha a^*, b\gamma b^*\delta a\alpha a^*) \in \xi, (b^*\delta b, b^*\delta b\gamma a^*\beta a) \in \xi$.

Proof: Let $a, b \in S$ be such that $(a,b) \in \rho$. Then there exist $a' \in V_{\alpha_1}^{\beta_1}(a), b' \in V_{\gamma_1}^{\delta_1}(b)$ such that $a'\beta_1 b \in K, (a\alpha_1 a', b\gamma_1 b'\delta a\alpha_1 a') \in \xi, (b'\delta_1 b, b'\delta_1 b\gamma_1 a'\beta_1 a) \in \xi$. Since ξ is an ip - congruence pair on $E(S)$. Now

$$\begin{aligned} a\alpha a^* &= (a\alpha a')\beta_1(a\alpha a^*) \\ &\;\xi\; (b\gamma_1 b')\delta(a\alpha_1 a')\beta_1(a\alpha a^*) \\ &= (b\gamma b^*)\delta(b\gamma_1 b')\delta(a\alpha_1 a')\beta_1(a\alpha a^*) \\ &\;\xi\; (b\gamma b^*)\delta(a\alpha_1 a')\beta_1(a\alpha a^*) \\ &= (b\gamma b^*)\delta(a\alpha a^*). \end{aligned}$$

and

$$
\begin{aligned}
b^*\delta b &= (b^*\delta b)\gamma_1(b'\delta_1 b) \\
&\ \xi\ (b^*\delta b)\gamma_1(b'\delta_1 b)\gamma_1(a'\beta_1 a) \\
&= (b^*\delta b)\gamma_1(b'\delta_1 b)\gamma_1(a'\beta_1 a)\gamma(a^*\delta_1 a) \\
&\ \xi\ (b^*\delta b)\gamma_1(b'\delta_1 b)\gamma(a^*\delta a) \\
&= (b^*\delta b)\gamma(a^*\delta a).
\end{aligned}
$$

To show that $a^*\beta b \in K$ notice that $a'\beta_1 b \in K$ and $(a\alpha_1 a', b\gamma_1 b'\delta_1 a\alpha_1 a') \in \xi$. Then by theorem 2.14 we have $a'\beta_1(a\alpha a^*)\beta b \in K$. Now since K is a full partial Γ-subsemigroup of S, we have

$$
\begin{aligned}
a^*\beta b &= a^*\beta a\alpha a^*\beta b \\
&= a^*\beta(a\alpha_1 a'\beta_1 a)\alpha a^*\beta b \\
&= (a^*\beta a)\alpha_1(a'\beta_1 a\alpha a^*\beta b) \in K.
\end{aligned}
$$

Theorem 2.16 Let (ξ, K) be an ip - congruence pair and $a, b \in S$ be such that for some $a' \in V_\alpha^\beta(a)$, $b' \in V_\gamma^\delta(b)$, $a'\alpha b \in K$, $(a\alpha a', b\gamma b'\delta a\alpha a') \in \xi$ and $(b'\delta b, b'\delta b\gamma a'\beta a) \in \xi$ then

(i) $(a'\beta a, a'\beta a\alpha b'\delta b) \in \xi$

(ii) $b'\delta a \in K$

(iii) $b\gamma a' \in K$

(iv) $(b\gamma b', a\alpha a'\beta b\gamma b') \in \xi$.

Proof: (i) Let $x \in RS(a\alpha a', b\gamma b')$; then $b'\delta x\beta a \in V_\gamma^\alpha(a'\beta b)$ and $a'\beta b\gamma b'\delta x\beta a = a'\beta x\beta a$. Now

$$
\begin{aligned}
a\alpha a'\beta x &= a\alpha a'\beta x\beta a\alpha a' \\
&\ \xi\ (a\alpha a')\beta x\beta(b\gamma b'\delta a\alpha a') \\
&= (a\alpha a')\beta(b\gamma b')\delta(a\alpha a') \\
&\ \xi\ (a\alpha a')\beta(a\alpha a') \\
&= a\alpha a'.
\end{aligned}
$$

and hence by normality of ξ we have

$$
\begin{aligned}
a'\beta x\beta a &= a'\beta(a\alpha a'\beta x)\beta a \\
&\ \xi\ a'\beta(a\alpha a')\beta a \\
&= a\beta a.
\end{aligned}
$$

Now by Theorem 2.14(ii) we have

$$\begin{aligned}
a'\beta a &= (a'\beta a)\alpha(a'\beta a)\alpha(a'\beta a) \\
&\xi\ (a'\beta a)\alpha(b'\delta x\beta a)\alpha(a'\beta a)\alpha(a'\beta b)\gamma(a'\beta a) \\
&= a'\beta a\alpha b'\delta(x\beta a\alpha a')\beta b\gamma a'\beta a \\
&= a'\beta a\alpha b'\delta x\beta b\gamma a'\beta a \\
&= a'\beta a\alpha b'\delta x\beta b\gamma(b'\delta b\gamma a'\beta a) \\
&\xi\ a'\beta a\alpha b'\delta x\beta b\gamma b'\delta b \\
&= a'\beta a\alpha b'\delta x\beta b.
\end{aligned}$$

and hence

$$\begin{aligned}
(a'\beta a)\alpha(b'\delta b) &\xi\ (a'\beta a\alpha b'\delta x\beta b)\gamma(b'\delta b) \\
&= a'\beta a\alpha b'\delta x\beta b \\
&\xi\ a'\beta a.
\end{aligned}$$

Hence $(a'\beta a, a'\beta a\alpha b'\delta b) \in \xi$.

(ii) Let $g \in RS(b\gamma b', a\alpha a')$. Then $a'\beta g\delta b \in V_\alpha^\gamma(b'\delta a)$. Now by theorem 2.14$(i)$ we have $a'\beta g\delta b \in K$ since $g \in E_\delta, a'\beta b \in K$ and $(a\alpha a', b\gamma b'\delta a\alpha a')\xi$. Since K is normal, we have $b'\delta a \in K$.

(iii) Now let $h \in RS(b'\delta b, a'\beta a)$, then $a\alpha h\gamma b' \in V_\beta^\delta(b\gamma a')$. Since $b'\delta a \in K$, K is normal and by (ii) we have $b\gamma b'\delta a\alpha a'\beta a\alpha h\gamma b' = b\gamma(b'\delta a\alpha h)\gamma b' \in K$ since $b'\delta a \in K$ and $h \in E(S) \subseteq K$. Now using (i) we have,

$$\begin{aligned}
h\gamma a'\beta a &= a'\beta a\alpha h\gamma a'\beta a \\
&\xi\ a'\beta a\alpha b'\delta b\gamma h\gamma a'\beta a \\
&= a'\beta a\alpha b'\delta b\gamma a'\beta a \\
&\xi\ a'\beta a\alpha a'\beta a \\
&= a'\beta a.
\end{aligned}$$

and we have

$$\begin{aligned}
a\alpha h\gamma b'\delta b\gamma a' &= a\alpha h\gamma b' \\
&= a\alpha h\gamma a'\beta a\alpha a' \\
&\xi\ a\alpha a'\beta a\alpha a' \\
&= a\alpha a' \\
&\xi\ b\gamma b'\delta a\alpha a'.
\end{aligned}$$

Hence by condition (A), $a\alpha h\gamma b' \in K$ and so $b\gamma a' \in K$ as required.

(iv) We now show that $(b\gamma b', a\alpha a'\beta b\gamma b') \in \xi$. Let $h \in RS(b'\delta b, a'\beta a)$. Then $a\alpha h\gamma b' \in V_\beta^\delta(b\gamma a')$. Since

$$\begin{aligned} b'\delta b\gamma h &= b'\delta b\gamma h\gamma b'\delta b \\ &\xi\ b'\delta b\gamma h\gamma b'\delta b\gamma a'\beta a \\ &= b'\delta b\gamma h\gamma a'\beta a \\ &= b'\delta b\gamma a'\beta a \\ &\xi\ b'\delta b. \end{aligned}$$

we have,

$$\begin{aligned} b\gamma a'\beta(a\alpha h\gamma b') &= b\gamma(a'\beta a\alpha h)\gamma b' \\ &= b\gamma h\gamma b' \\ &= b\gamma(b'\delta b\gamma h)\gamma b' \\ &\xi\ b\gamma b'\delta b\gamma b' \\ &= b\gamma b'. \end{aligned}$$

Now by (iii) $b\gamma a' \in K$ and hence by theorem 2.14(ii), we have

$$\begin{aligned} b\gamma b' &= (b\gamma b')\delta(b\gamma b')\delta(b\gamma b') \\ &\xi\ (b\gamma b')\delta(a\alpha h\gamma b')\delta(b\gamma b')\delta(b\gamma a')\beta(b\gamma b') \\ &= b\gamma b'\delta a\alpha h\gamma a'\beta b\gamma b' \\ &= b\gamma b'\delta a\alpha a'\beta a\alpha h\gamma a'\beta b\gamma b' \\ &= (b\gamma b'\delta a\alpha a')\beta(a\alpha h\gamma a'\beta b\gamma b') \\ &\xi\ a\alpha a'\beta a\alpha h\gamma a'\beta b\gamma b' \\ &= a\alpha h\gamma a'\beta b\gamma b'. \end{aligned}$$

Hence

$$\begin{aligned} a\alpha a'\beta b\gamma b' &\ \xi\ (a\alpha a')\beta(a\alpha h\gamma a'\beta b\gamma b') \\ &= a\alpha h\gamma a'\beta b\gamma b' \\ &\xi\ b\gamma b'. \end{aligned}$$

Thus $(b\gamma b', a\alpha a'\beta b\gamma b') \in \xi$. Hence the theorem.

Theorem 2.17 Let (ξ, K) be an ip - congruence pair for S. Then $\rho = \rho_{(\xi,K)}$ is an equivalence relation on S.

Proof: Let (ξ, K) be an ip - congruence pair for S. Then ρ is clearly reflexive since $E(S) \subseteq K$ and ξ is reflexive. The symmetry of ρ follows from $(i), (ii)$ and (iv) of theorem 3.13. To show that ρ is transitive, let $(a, b) \in \rho$ and $(b, c) \in \rho$. Also let $a' \in V_\alpha^\beta(a), b' \in V_\gamma^\delta(b)$ and $c' \in V_\mu^\nu(c)$. Then by definition of ρ we have $(a\alpha a', b\gamma b'\delta a\alpha a') \in \xi, (b'\delta b, b'\delta b\gamma a'\beta a) \in \xi, a'\beta b \in K, (b\gamma b', c\mu c'\nu b\gamma b') \in \xi(c'\nu c, c'\nu c\mu b'\delta b) \in \xi, b'\delta c \in K$. Since ξ is compatible and transitive we have,

$$\begin{aligned} a\alpha a' \;&\xi\; b\gamma b'\delta a\alpha a' \\ &\xi\; c\mu c'\nu b\gamma b'\delta a\alpha a' \\ &\xi\; c\mu c'\nu a\alpha a'. \end{aligned}$$

and

$$\begin{aligned} c'\nu c\mu a'\beta a \;&\xi\; c'\nu c\mu b'\delta b\gamma a'\beta a \\ &\xi\; c'\nu c\mu b'\delta b \\ &\xi\; c'\nu c. \end{aligned}$$

Hence $(a\alpha a', c\mu c'\nu a\alpha a') \in \xi$ and $(c'\nu c, c'\nu c\mu a'\beta a) \in \xi$. On the other hand, by symmetry of ρ, we have $b'\delta a \in K, c'\nu b \in K$ and $(b\gamma b', a\alpha a'\beta b\gamma b') \in \xi$. Let $g \in RS(c\mu c', b\gamma b')$ then $b'\delta g\nu a \in K$ by theorem 2.14(i). Therefore

$$(c'\nu b)\gamma(b'\delta g\nu a) = (c'\nu b)\gamma(b'\delta g\nu c)\mu c'\nu a \in K \tag{1}$$

where $b'\delta g\nu c \in V_\gamma^\mu(c'\nu b)$. Again let $h \in RS(c\mu c', a\alpha a')$ then $a'\beta h\nu c \in V_\alpha^\mu(c'\nu a)$. Since $(b, c) \in \rho$ and ρ is symmetric, we have

$$\begin{aligned} c\mu c' \;&=\; c\mu c'\nu c\mu c' \\ &\xi\; c\mu c'\nu b\gamma b'\delta c\mu c' \\ &=\; c\mu c'\nu g\nu b\gamma b'\delta c\mu c' \\ &\xi\; c\mu c'\nu g\nu c\mu c' \\ &=\; c\mu c'\nu g. \end{aligned}$$

and since ξ is normal, we have

$$c'\nu c\xi c'\nu g\nu c = c'\nu b\gamma b'\delta g\nu c \tag{2}$$

Again since $(b\gamma b', a\alpha a'\beta b\gamma b') \in \xi$ and $(c\mu c', b\gamma b'\delta c\mu c') \in \xi$ we have

$$\begin{aligned} c\mu c' &\ \xi\ b\gamma b'\delta c\mu c' \\ &\ \xi\ a\alpha a'\beta b\gamma b'\delta c\mu c' \\ &\ \xi\ a\alpha a'\beta c\mu c' \end{aligned}$$

and hence similarly we can show that

$$c\mu c'\xi c'\nu h\nu c = c'\nu a\alpha a'\beta h\nu c \tag{3}$$

Thus we have from (1) and (2) $(c'\nu b\gamma b'\delta g\nu c, c'\nu a\alpha a'\beta h\nu c) \in \xi$. Now from (1), (2) and (3) and by condition (A) we have $c'\nu a \in K$ and hence $(a,c) \in \rho$ which completes the proof.

Theorem 2.18 Let (ξ, K) be an ip - congruence pair on a right orthodox Γ-semigroup S and $a' \in V_{\alpha_1}^{\beta_1}(a), b' \in V_{\alpha_2}^{\beta_2}(b), c' \in V_{\alpha_3}^{\beta_3}(c)$ then
(i) $a \in K$ implies $(a\alpha_1 a')\xi \in V_{\alpha_1}^{\beta_1}((a'\beta_1 a)\xi)$.
(ii) $a \in K, x \in RS(a'\beta_1 a, a\alpha_1 a')$ implies that $(a\alpha_1 x\alpha_1 a', a\alpha_1 a') \in \xi$ and $(a'\beta_1 x\alpha_1 a, a'\beta_1 a) \in \xi$.
(iii) $(a,b) \in \rho, g \in RS(c'\beta_3 c, a\alpha_1 a'), h \in RS(c'\beta_3 c, b\alpha_2 b')$ implies that
$(b'\beta_2 h\alpha_3 b\ ,\ b'\beta_3 h\alpha_3 b\alpha_2 a'\beta_1 g\alpha_3 a) \in \xi$.
(iv) $(a,b)\rho, g \in RS(a'\beta_1 a, c\alpha_3 c'), h \in RS(b'\beta_2 b, c\alpha_3 c')$ implies that
$(a\alpha_1 g\alpha_1 a', b\alpha_2 h\alpha_2 b'\beta_2 a\alpha_1 g\alpha_1 a') \in \xi$.

Proof: (i) If $a \in K$ then by theorem 2.14(ii)

$$(a\alpha_1 a'\beta_1 a\alpha_1 a'\beta_1 a\alpha_1 a', (a\alpha_1 a')\beta_1 a'\beta_1(a\alpha_1 a'\beta_1 a\alpha_1(a\alpha_1 a')) \in \xi$$

i.e, $(a\alpha_1 a', (a\alpha_1 a')\beta_1(a'\beta_1 a)\alpha_1(a\alpha_1 a')) \in \xi$. Again since K is a totally regular Γ-subsemigroup of S we have $a' \in K$ and by similar argument we have $(a'\beta_1 a, (a'\beta_1 a)\alpha_1(a\alpha_1 a')\beta_1(a'\beta_1 a)) \in \xi$. Hence we have $(a\alpha_1 a')\xi \in V_{\alpha_1}^{\beta_1}((a'\beta_1 a)\xi)$.
(ii) Since $a \in K$ and $x \in RS(a'\beta_1 a, a\alpha_1 a')$, by theorem 2.14$(ii)$ we have

$$((a\alpha_1 a')\beta_1(a\alpha_1 x\alpha_1 a')\beta_1(a\alpha_1 a'), (a\alpha_1 a')\beta_1 a'\beta_1(a\alpha_1 x\alpha_1 a')\beta_1 a\alpha_1(a\alpha_1 a')) \in \xi$$

i.e, $(a\alpha_1 x\alpha_1 a', (a\alpha_1 a')\beta_1(a'\beta_1 a)\alpha_1(a\alpha_1 a')) \in \xi$. Hence using (i) we have $(a\alpha_1 x\alpha_1 a', a\alpha_1 a') \in \xi$. Similarly we can show that $(a'\beta_1 x\alpha_1 a, a'\beta_1 a) \in \xi$ by using $a' \in K$.
(iii) Let $(a,b) \in \rho$. Since ρ is symmetric we have $(b,a) \in \rho$. From theorem 3.12(iii) we have $a\alpha_1 b' \in K, ((a\alpha_1 a')\xi, (b\alpha_2 b')\xi) \in \mathcal{R}$ and $((a'\beta_1 a)\xi, (b'\beta_2 b)\xi) \in \mathcal{L}$ in $E(S)/\xi$. Since $g \in RS(c'\beta_3 c, a\alpha_1 a'), h \in RS(c'\beta_3 c, b\alpha_2 b'), x \in RS(a'\beta_1 a, b'\beta_2 b)$, we have $a'\beta_1 g\alpha_3 c' \in V_{\alpha_1}^{\beta_3}(c\alpha_3 a)$,

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$b'\beta_2 h\alpha_3 c' \in V_{\alpha_2}^{\beta_3}(c\alpha_3 b)$ and $b\alpha_2 x\alpha_1 a' \in V_{\beta_2}^{\beta_1}(a\alpha_1 b')$. Again let $t \in RS(g, (a\alpha_1 b')\beta_2(b\alpha_2 x\alpha_1 a')) = RS(g, a\alpha_1 x\alpha_1 a')$. Hence we have $(b\alpha_2 a\alpha_1 a')\beta_1 t\alpha_3 g \in V_{\beta_2}^{\alpha_3}(g\alpha_3 a\alpha_1 b')$ and $b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b' = b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 g\alpha_3(g\alpha_3 a\alpha_1 b') \in E_{\beta_2}$. Again since ξ is compatible and $(a'\beta_1 a, a'\beta_1 a\alpha_1 b'\beta_2 b) \in \xi$,

$$\begin{aligned} a'\beta_1 a\alpha_1 x &= a'\beta_1 a\alpha_1 x\alpha_1 a'\beta_1 a \\ &\ \xi\ a'\beta_1 a\alpha_1 x\alpha_1 a'\beta_1 a\alpha_1 b'\beta_2 b \\ &= a'\beta_1 a\alpha_1 b'\beta_2 b \\ &\ \xi\ a'\beta_1 a. \end{aligned}$$

Hence by normality of ξ we have

$$(a\alpha_1 x\alpha_1 a', a\alpha_1 a') \in \xi \tag{4}$$

Also $(a\alpha_1 b')\beta_2(b\alpha_2 x\alpha_1 a') = a\alpha_1 x\alpha_1 a'\xi a\alpha_1 a'$ and $a\alpha_1 b' \in K$, so by theorem 2.14(ii) we get $(a\alpha_1 a')\beta_1 t\alpha_3(a\alpha_1 a')\xi(a\alpha_1 a')\beta_1(b\alpha_2 x\alpha_1 a')\beta_1 t\alpha_3(a\alpha_1 b')\beta_2(a\alpha_1 a')$.

Now

$$\begin{aligned} a\alpha_1 a'\beta_1 t &= (a\alpha_1 a')\beta_1(a\alpha_1 x\alpha_1 a')\beta_1 t \\ &= (a\alpha_1 x\alpha_1 a')\beta_1 t \\ &= t. \end{aligned}$$

i.e,

$$a\alpha_1 a'\beta_1 t = t \tag{5}$$

Now since $(a\alpha_1 a'\beta_1 b\alpha_2 b', b\alpha_2 b') \in \xi$ we have

$$\begin{aligned} t\alpha_3 a\alpha_1 a' &\ \xi\ (a\alpha_1 a'\beta_1 b\alpha_2 b')\beta_2(b\alpha_2 x\alpha_1 a')\beta_1 t\alpha_3(a\alpha_1 b')\beta_2(a\alpha_1 a') \\ &\ \xi\ b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3(a\alpha_1 b')\beta_2(a\alpha_1 a'). \end{aligned}$$

Thus

$$\begin{aligned} t\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' &\ \xi\ (b\alpha_2 x\alpha_1 a')\beta_1 t\alpha_3(a\alpha_1 b')\beta_1(b\alpha_2 b') \\ &\ \xi\ (b\alpha_2 x\alpha_1 a')\beta_1 t\alpha_3(a\alpha_1 b')\beta_2(b\alpha_2 b') \\ &= (b\alpha_2 x\alpha_1 a')\beta_1 t\alpha_3(a\alpha_1 b'. \end{aligned}$$

i.e,

$$(t\alpha_3 b\alpha_2 b', (b\alpha_2 x\alpha_1 a')\beta_1 t\alpha_3 (a\alpha_1 b') \in \xi \tag{6}$$

Now since $a\alpha_1 a'\beta_1 t = t$ and $a'\beta_1 t\alpha_3 a \in E_{\alpha_1}$, we have

$$\begin{aligned}
b'\beta_2 b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b'\beta_2 b &= b'\beta_2 b\alpha_2 x\alpha_1 a'\beta_1 a\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 a'\beta_1 a\alpha_1 b'\beta_2 b \\
&\xi\ b'\beta_2 b\alpha_2 a'\beta_1 a\alpha_2 x\alpha_1 a'\beta_1 a\alpha_1 b'\beta_2 b\alpha_2 a'\beta_1 t\alpha_3 a\alpha_1 a'\beta_1 a \\
&= b'\beta_2 b\alpha_2 a'\beta_1 a\alpha_2 x\alpha_1 b'\beta_2 b\alpha_2 a'\beta_1 t\alpha_3 a \\
&= b'\beta_2 b\alpha_2 a'\beta_1 a\alpha_2 b'\beta_2 b\alpha_2 a'\beta_1 t\alpha_3 a \\
&\xi\ b'\beta_2 b\alpha_2 a'\beta_1 t\alpha_3 a.
\end{aligned}$$

Thus

$$(b'\beta_2 b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b'\beta_2 b, b'\beta_2 b\alpha_2 a'\beta_1 t\alpha_3 a) \in \xi \tag{7}$$

Again

$$\begin{aligned}
h\alpha_3 t\alpha_3 b\alpha_2 b' &= h\alpha_3 c'\beta_3 c\alpha_3 t\alpha_3 b\alpha_2 b' \\
&= h\alpha_3 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 t\alpha_3 b\alpha_2 b' \text{ (by (5))} \\
&= h\alpha_3 c'\beta_3 c\alpha_3 g\alpha_3 a\alpha_1 a'\beta_1 t\alpha_3 b\alpha_2 b' \\
&= h\alpha_3 c'\beta_3 c\alpha_3 g\alpha_3 t\alpha_3 b\alpha_2 b' \\
&\xi\ h\alpha_3 c'\beta_3 c\alpha_3 g\alpha_3 t\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (since } (b,a) \in \rho) \\
&\xi\ h\alpha_3 c'\beta_3 c\alpha_3 g\alpha_3 t\alpha_3 a\alpha_1 x\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (By (4))} \\
&= h\alpha_3 c'\beta_3 c\alpha_3 g\alpha_3 a\alpha_1 x\alpha_1 a'\beta_1 b\alpha_2 b' \\
&\xi\ h\alpha_3 c'\beta_3 c\alpha_3 g\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \\
&= h\alpha_3 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \\
&= h\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \\
&= h\alpha_3 b\alpha_2 b'
\end{aligned}$$

Thus

$$(h\alpha_3 t\alpha_3 b\alpha_2 b', h\alpha_3 b\alpha_2 b') \in \xi \tag{8}$$

Now from (6) and (8) we have

$$(h\alpha_3 b\alpha_2 b', h\alpha_3 (b\alpha_2 x\alpha_1 a')\beta_1 t\alpha_3 (a\alpha_1 b')) \in \xi$$

Since ξ is normal and $b'\beta_2 h\alpha_3 b \in E_{\alpha_2}$, we have

$$(b'\beta_2 h\alpha_3 b)\alpha_2 x\alpha_1(a'\beta_1 t\alpha_3 a)\alpha_1(b'\beta_2 b) \in E_{\alpha_2}$$

Hence

$$(b'\beta_2 h\alpha_3 b, b'\beta_2 h\alpha_3 b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b'\beta_2 b) \in \xi$$

Also from (7) we have

$$\begin{aligned} b'\beta_2 h\alpha_3 b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b'\beta_2 b &= (b'\beta_2 h\alpha_3 b)\alpha_2(b'\beta_2 b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b'\beta_2 b) \\ &\ \xi\ (b'\beta_2 h\alpha_3 b)\alpha_2(b'\beta_2 b\alpha_2 a'\beta_1 t\alpha_3 a) \\ &= b'\beta_2 h\alpha_3 b\alpha_2 a'\beta_1 t\alpha_3 a. \end{aligned}$$

Thus

$$(b'\beta_2 h\alpha_3 b, b'\beta_2 h\alpha_3 b\alpha_2 a'\beta_1 t\alpha_3 a) \in \xi$$

Since $a\alpha_1 a'\beta_1 g = g$ and $t\alpha_3 g = t$ we have

$$\begin{aligned} b'\beta_2 h\alpha_3 b\alpha_2 a'\beta_1 g\alpha_3 a &\ \xi\ b'\beta_2 h\alpha_3 b\alpha_2 a'\beta_1 t\alpha_3 a\alpha_1 a'\beta_1 g\alpha_3 a \\ &= b'\beta_2 h\alpha_3 b\alpha_2 a'\beta_1 t\alpha_3 g\alpha_3 a \\ &= b'\beta_2 h\alpha_3 b\alpha_2 a'\beta_1 t\alpha_3 a \\ &\ \xi\ b'\beta_2 h\alpha_3 b. \end{aligned}$$

Hence $(b'\beta_2 h\alpha_3 b b'\beta_2 h\alpha_3 b\alpha_2 a'\beta_1 g\alpha_3 a) \in \xi$. Which completes the proof.

(iv) is similar to (iii).

Theorem 2.19 Let (ξ, K) be an ip - congruence pair of S. Then ρ is an ip - congruence on S.

Proof: Let (ξ, K) be a congruence pair of S. Let $a, b, c \in S$ be such that $(a, b) \in \rho$. Then for any $a' \in V_{\alpha_1}^{\beta_1}(a), b' \in V_{\alpha_2}^{\beta_2}(b)$ we have $a'\beta_1 b \in K$, $(a\alpha_1 a', b\alpha_2 b'\beta_2 a\alpha_1 a') \in \xi$ and $(b'\beta_2 b, b'\beta_2 b\alpha_2 a'\beta_1 a) \in \xi$. To show that ρ is an ip - congruence on S we are to show only $(c\alpha_3 a, c\alpha_3 b) \in \rho$ and $(a\alpha_1 c, b\alpha_2 c) \in \rho$ for any $c \in S$ with $V_{\alpha_3}^{\beta_3}(c) \neq \phi$. Let $c' \in V_{\alpha_3}^{\beta_3}(c)$. Again let $g \in RS(c'\beta_3 c, a\alpha_1 a')$ and $h \in RS(c'\beta_3 c, b\alpha_2 b')$. Then $a'\beta_1 g\alpha_3 c' \in V_{\alpha_1}^{\beta_3}(c\alpha_3 a)$ and $b'\beta_2 h\alpha_3 c' \in V_{\alpha_2}^{\beta_3}(c\alpha_3 b)$. Also $(c\alpha_3 a)\alpha_1(a'\beta_1 g\alpha_3 c') = c\alpha_3 g\alpha_3 c'$ and $(c\alpha_3 b)\alpha_2(b'\beta_2 h\alpha_3 c') = c\alpha_3 h\alpha_3 c'$. Now $(a, b) \in \rho$ and hence

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$$\begin{aligned}
c'\beta_3 c\alpha_3 g &= c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 g \\
&\ \xi\ (c'\beta_3 c)\alpha_3 (b\alpha_3 b'\beta_3 a\alpha_1 a')\beta_1 g \\
&= (c'\beta_3 c)\alpha_3 h\alpha_3 (b\alpha_3 b')\beta_3 (a\alpha_1 a')\beta_1 g \\
&\ \xi\ (c'\beta_3 c)\alpha_3 h\alpha_3 (a\alpha_1 a')\beta_1 g \\
&= (c'\beta_3 c)\alpha_3 h\alpha_3 g.
\end{aligned}$$

Again $c\alpha_3 h\alpha_3 c'$ and $c\alpha_3 h\alpha_3 g\alpha_3 c'$ are in E_{β_3}, as S is a right orthodox Γ-semigroup and hence by normality of ξ we have

$$\begin{aligned}
c\alpha_3 g\alpha_3 c' &\ \xi\ c\alpha_3 (c'\beta_3 c\alpha_3 g)\alpha_3 c' \\
&\ \xi\ c\alpha_3 (c'\beta_3 c\alpha_3 h\alpha_3 g)\alpha_3 c' \\
&= c\alpha_3 h\alpha_3 g\alpha_3 c' \\
&= (c\alpha_3 h\alpha_3 c')\beta_3 (c\alpha_3 g\alpha_3 c').
\end{aligned}$$

i.e,

$$(c\alpha_3 g\alpha_3 c'\, ,\, (c\alpha_3 h\alpha_3 c')\beta_3 (c\alpha_3 g\alpha_3 c')) \in \xi \tag{9}$$

Now $b'\beta_2 h\alpha_3 b = (b'\beta_2 h\alpha_3 c')\beta_3 (c\alpha_3 b)$ and $a'\beta_1 g\alpha_3 a = (a'\beta_1 g\alpha_3 c')\beta_3 (c\alpha_3 a$ and by Theorem 2.16(iii) we have

$$(b'\beta_2 h\alpha_3 b\, ,\, b'\beta_2 h\alpha_3 b)\alpha_2 (a'\beta_1 g\alpha_3 a)) \in \xi \tag{10}$$

Also by Theorem 2.14(i) we have

$$(a'\beta_1 g\alpha_3 c')\beta_3 (c\alpha_3 b) = a'\beta_1 (g\alpha_3 c'\beta_3 c)\alpha_3 b = a'\beta_1 g\alpha_3 b \in K \tag{11}$$

Since $a'\beta_1 b \in K$ and $(a\alpha_1 a'\, ,\, b\alpha_2 b'\beta_2 a\alpha_1 a') \in \xi$.

From (9),(10) and (11) we have

$$(c\alpha_3 a\, ,\, c\alpha_3 b) \in \xi \tag{12}$$

We now show that $(a\alpha_1 c\,,\, b\alpha_2 c) \in \xi$. For this let $g \in RS(a'\beta_1 a\,,\, c\alpha_3 c')$ and $h \in RS(b'\beta_2 b\,,\, c\alpha_3 c'$. Then $c'\beta_3 g\alpha_1 a' \in V^{\beta_2}_{\alpha_3}(b\alpha_2 c)$. Now since K is a full self conjugate Γ-subsemigroup of S and $a'\beta_1 b \in K$, we have

$$(c'\beta_3 g\alpha_1 a')\beta_1(b\alpha_2 c) = c'\beta_3(g\alpha_1(a'\beta_1 b))\alpha_2 c \in K \tag{13}$$

Now $(a\alpha_1 c)\alpha_3(c'\beta_3 g\alpha_1 a') = a\alpha_1 g\alpha_1 a'$ and $(b\alpha_2 h\alpha_2 b'$ and from 4(ii) we have

$$(a\alpha_1 g\alpha_1 a', (b\alpha_2 h\alpha_2 b')\beta_2(a\alpha_1 g\alpha_1 a')) \in \xi \tag{14}$$

Again $(c'\beta_3 g\alpha_1 a')\beta_1(a\alpha_1 c) = c'\beta_3 g\alpha_1 c$ and $(c'\beta_3 h\alpha_2 b')\beta_2(b\alpha_2 c) = c'\beta_3 h\alpha_2 c$. Now

$$\begin{aligned}
h\alpha_2 c\alpha_3 c' &= h\alpha_2 b'\beta_2 b\alpha_2 c\alpha_3 c' \\
&\ \xi\ h\alpha_2(b'\beta_2 b\alpha_2 a'\beta_1 a)\alpha_1 c\alpha_3 c' \\
&= (h\alpha_2 b'\beta_2 b)\alpha_2(a'\beta_1 a\alpha_1 g\alpha_1 c\alpha_3 c') \\
&\ \xi\ h\alpha_2 b'\beta_2 b\alpha_2 g\alpha_1 c\alpha_3 c' \\
&= h\alpha_2 g\alpha_1 c\alpha_3 c'.
\end{aligned}$$

Now by normality of ξ we have $c'\beta_3 h\alpha_2 c\,\xi\, c'\beta_3 h\alpha_2 g\alpha_1 c = (c'\beta_3 h\alpha_2 c)\alpha_3(c'\beta_3 g\alpha_1 c)$. Thus

$$(c'\beta_3 h\alpha_2 c\,,\ (c'\beta_3 h\alpha_2 c)\alpha_3(c'\beta_3 g\alpha_1 c)) \in \xi \tag{15}$$

From (13),(14) and (15) we have $(a\alpha_1 c\,,\, b\alpha_2 c) \in \xi$. Hence ρ is an equivalence relation by Theorem (4) ρ is an ip - congruence on S.

Theorem 2.20 Let S be a right orthodox Γ-semigroup. If (ξ, K) is an ip - congruence pair for S then $\rho_{(\xi,K)}$ is an ip - congruence on S with trace ξ and Kernel K. Conversely, if ρ is an ip - congruence on S then $(tr\rho, Ker\rho)$ is an ip - congruence pair for S and $\rho = \rho_{(tr\rho, Ker\rho)}$

Proof:

Let (ξ, K) be an ip - congruence pair for S. Then by theorem 2.17 $\rho = \rho_{(\xi,K)}$ is an ip - congruence on S. We now show that $tr\rho = \xi$ and $Ker\rho = K$.

Let e be an α-idempotent and f be a β-idempotent and let $(e, f) \in \rho$. Since $e \in V^{\alpha}_{\alpha}(e)$ and $f \in V^{\beta}_{\beta}(f)$ we have $(e, f\beta e) \in \xi$ and $(f, f\beta e) \in \xi$. Since ξ is an equivalence relation we have $(e, f) \in \xi$. i.e, $tr\rho \subseteq \xi$. Again suppose that for $e \in E_\alpha$ and $f \in E_\beta, (e, f) \in \xi$. Let $x \in V_{\alpha_1}^{\beta_1}(e)$ and $y \in V_{\alpha_2}^{\beta_2}(f)$. Since $e \in K$ and K is regular, $x \in K$ and hence $x\beta_1 f \in K$ since K is a Γ-subsemigroup. Again

$$\begin{aligned} y\beta_2 f &= y\beta_2 f\beta f \\ &\ \xi\ (y\beta_2 f)\beta e \\ &\ \xi\ (y\beta_2 f)\beta(e\alpha_1 x\beta_1 e) \\ &\ \xi\ (y\beta_2 f)\beta f\alpha_2 x\beta_1 e \\ &\ \xi\ (y\beta_2 f)\alpha_2(x\beta_1 e). \end{aligned}$$

and

$$\begin{aligned} e\alpha_1 x &= e\alpha e\alpha_1 x \\ &\ \xi\ f\alpha_2(e\alpha_1 x) \\ &= (f\alpha_2 y\beta_2 f)\alpha_2(e\alpha_1 x) \\ &= (f\alpha_2 y)\beta_2(f\alpha_2 e\alpha_1 x) \\ &\ \xi\ (f\alpha_2 y)\beta_2(e\alpha e\alpha_1 x) \\ &= (f\alpha_2 y)\beta_2(e\alpha_1 x). \end{aligned}$$

Hence $x\beta_1 f \in K$ and $(e\alpha_1 x, (f\alpha_2 y)\beta_2(e\alpha_1 x)) \in \xi$ and $(y\beta_2 f, (y\beta_2 f)\alpha_2(x\beta_1 e)) \in \xi$. Thus $(e, f) \in \rho$ and hence $\xi \subseteq tr\rho$. i.e, $tr\rho = \xi$.

Let us now show that $Ker\rho = K$. Let $a \in Ker\rho$. Then there exists $e \in E_\alpha$ such that $(a, e) \in \rho$. Hence we have $a'\delta e \in K$ and $(a\gamma a', e\alpha e\gamma a') \in \xi$ for any $a' \in V_\gamma^\delta(a)$. Hence by Theorem 2.16(ii) we have $e\alpha a \in K$. Again since $e\alpha a = (e\alpha a\gamma a')\delta a$, by condition (A) we have $a \in K$. Thus $Ker\rho \subseteq K$. Conversely let $a \in K$. Let $a' \in V_\alpha^\beta(a)$ and $x \in RS(a'\beta a, a\alpha a')$ then $a'\beta x\alpha a' \in V_\alpha^\beta(a\alpha a)$. Hence by Theorem 2.18(ii) we have $a\alpha a'\ \xi\ a\alpha x\alpha a' = a\alpha(a\alpha a'\beta x\alpha a') = (a\alpha a)\alpha(a'\beta x\alpha a')$ and $a'\beta a\ \xi\ a'\beta x\alpha a = a'\beta x\alpha a'\beta a\alpha a = (a'\beta x\alpha a')\beta(a\alpha a)$. Again $a \in K$ implies $a' \in K$ and so $a'\beta a\alpha a \in K$ since K is totally regular Γ-subsemigroup of S. Hence we have $(a, a\alpha a) \in \rho$. This implies $a \in Ker\rho$. Hence $K = Ker\rho$.

Conversely suppose that ρ is an ip - congruence. Then $tr\rho$ is a normal ip - congruence on $E(S)$ and $Ker\rho$ is a normal partial Γ-subsemigroup of S. Let us now show that $(tr\rho, \ker\rho)$ satisfies conditions (A) and (B). To show (A), let for $a \in S$, $a' \in V_\alpha^\beta(a)$ and $e \in E_\gamma$, $e\gamma a \in Ker\rho$ and $(e, a\alpha a') \in tr\rho$. Now

$$\begin{aligned} a &= a\alpha a'\beta a \\ &= (a\alpha a')\beta a \\ &\rho\ e\gamma a (\text{since } \rho \text{ is an ip - congruence }) \\ &\rho\ f. \end{aligned}$$

for some δ-idempotent f. Hence $a \in Ker\rho$. Thus the condition (A) holds.

Next let $(a, f) \in \rho$ for some δ-idempotent f. Since $a \in Ker\rho, a' \in ker\rho$ for every $a' \in V_\alpha^\beta(a)$. Let $(a', g) \in \rho$ for some μ-idempotent g. Now

$$\begin{aligned} a'\beta a'\beta e\gamma a\alpha a &\ \rho\ g\mu g\mu e\gamma f\delta f \\ &= g\mu e\gamma f \\ &\ \rho\ a'\beta e\gamma a. \end{aligned}$$

Thus the condition (B) holds.

We now show that $\rho = \rho_{(tr\rho, Ker\rho)}$. Let $(a, b) \in \rho$ and $a' \in V_\alpha^\beta(a)$ and $b' \in V_\gamma^\delta(b)$. Now since ρ is an ip - congruence $a'\beta b\rho a'\beta a$. Hence $a'\beta b \in Ker\rho$. Again since $(a, b) \in \rho$ and ρ is an ip - congruence on S we have

$$\begin{aligned} a\alpha a' &\ \rho\ b\gamma a' \\ &= b\gamma b'\delta b\gamma a' \\ &\ \rho\ b\gamma b'\delta a\alpha a'. \end{aligned}$$

and

$$\begin{aligned} b'\delta b &\ \rho\ b'\delta a \\ &= b'\delta a\alpha a'\beta a \\ &\ \rho\ b'\delta b\gamma a'\beta a. \end{aligned}$$

Thus we can say that $\rho \subseteq \rho_{(tr\rho, Ker\rho)}$. Converse case follows from Theorem 2.12. Hence the proof.

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