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Continuous Solutions of a Quadratic Integral Equation

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Abstract: An existence theorem for a quadratic integral equation is proved by using Darbo fixed point theorem via a measure of noncompact -ness. The solution will be in the class $C(I)$ of continuous functions on the interval $I = [0, M]$. Finally, the existence of a fractional integral equation will be investigated.

Keywords: *Quadratic integral equation, Hausdorff measure of noncompactness, Modulus of continuity, Superposition operator, Darbo fixed point theorem.*

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1- Introduction. Due to the great importance of the integral equations for many scientific branches such as physics, engineering, economics and biology [5,6,7,8], we discuss a certain kind of the class of integral equations, that is the class of the quadratic integral equations, which take the form:

$$x(t) = g(t) + (Tx)(t) \int_0^t k(t,s) f\big(s, x(s)\big) ds, \quad t \in I \quad (1)$$

This kind of integral equations are inserted in the theories of radiative transfer and neutron transport and in the kinetic theory of gases [6,13].

This equation is a general form of another equation that was investigated in [3, 10], another type of this equation was treated in the class of monotonic functions [12] .

The goal of this paper is to prove the existence theorem of equation (1) in the class $C(I)$ of functions defined and continuous on the interval $I = [0, M]$.

2- Preliminaries. To perform our main theorem, first let E be a Banach space with a norm $\|.\|$ and θ its zero vector. Denote by B_r the closed ball in E centered at θ and its radius r .

The modulus of continuity $\omega(x, \varepsilon)$ of a function $x \in X, X$ is a nonempty bounded subset of the class $C(I)$, ε>0 is defined as [2]:

$$\omega(x, \varepsilon) = sup\{|x(t) - x(s)|: t, s \in I, |t - s| \leq \varepsilon\} \qquad (2)$$

From this definition, we can see that $\omega(x, \varepsilon) < \varepsilon'$ if the function $x(t)$ is continuous on I .

Next, let us put

$$\omega_0(X) = \lim_{\varepsilon \to 0} sup\{\omega(x, \varepsilon): x \epsilon X\}$$

For our benefit, we can consider the case in which the Banach space E is the space $C(I)$, with standard norm

$$\|x\| = max\{|x(t)|: t \epsilon I\}$$

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021 - 030

Further, let us define the following quantities [12]:

$$d(x) = \sup\{\, |x(s) - x(t)| - [\, x(s) - x(t)\,]: t, s \in I, t \leq s\}, \qquad (3)$$

and

$$d(X) = \sup\{\, d(x): x \in X \,\}$$

Notice that $d(X) = 0$ if and only if all functions belonging to X are nondecreasing on .

Now , let us define the function μ by putting

$$\mu(x) = \omega_0(X) + d(X).$$

It can be proved that the function μ is a measure of noncompactness in the space $C(I)$ [4].

Next, we will quote Darbo fixed point theorem [9]:

Theorem (1). Let Q be a nonempty, bounded, closed and convex subset of the Banach space E and let $A: Q \rightarrow Q$ be a continuous operator such that $\mu(AX) \leq c\mu(X)$ for any nonempty subset X of Q, $c\epsilon[0,1]$ is a constant, where μ is a measure of noncompactness, then A has at least one fixed point in Q .

In the sequence, we will define the superposition operator

$$(Fx)(t) = f\big(t, x(t)\big) \qquad (4)$$

generated by the function

$$f = f(t, x): [0,1] \times R \rightarrow R$$

and we have the following theorem [1]:

Theorem (2). The superposition operator F maps continuously the space $C(I)$ into itself iff f is continuous on $I \times R$.

In the sequel, we define the linear integral operator

$$(Kx)(t) = \int_0^t k(t, s) x(s) ds, \qquad (5)$$

where

$$k = k(t, s): I \times I \rightarrow R$$

and we will prove the following lemma:

ISSN: 2349-7955
021 - 030

Lemma (3). If $k = k(t,s): I \times I \rightarrow R$ is continuous for both two variables t and s , then the linear operator K , defined by (5), maps continuously the space $C(I)$ into itself.

Proof:

For $\varepsilon > 0$, assume that $\|x - y\| < \varepsilon$, then we have

$$|(Kx)(t) - (Ky)(t)| \leq \int_0^t |k(t,s)|\,|x(s) - y(s)|ds < \varepsilon \int_0^t |k(t,s)|\,ds$$

Since $k(t,s)$ is continuous on $I \times I$, then it is bounded and so the continuity of K is proved.

Next, if $x \epsilon C(I)$, then for $\varepsilon > 0$, such that $|t_2 - t_1| < \varepsilon, t_2 > t_1$ belong to I, we have

$$|(Kx)(t_2) - (Kx)(t_1)| < \int_0^{t_2} |k(t_2,s) - k(t_1,s)|\,|x(s)|ds + \qquad + \int_{t_1}^{t_2} |k(t_1,s)|\,|x(s)|ds$$

Due to the continuity of k and x , we deduce that $Kx \epsilon C(I)$.

3- Main Result. This section is devoted to discuss the solvability of the integral equation (1) in the space $C(I)$. For our purposes, we assume that

$$(Hx)(t) = g(t) + (Tx)(t) \int_0^t k(t,s) f\big(s, x(s)\big) ds, \qquad t \in I = [0, M]$$

Then equation (1) becomes

$$x = Hx = g + (Tx)KF(x),$$

Where F is the superposition operator generated by the function f and K is the linear integral operator generated by the kernel $k(t,s)$ defined above by (4) and (5) respectively.

We will investigate the integral equation (1) under the following

assumptions:

(i) the function g is a nondecreasing , nonnegative and continuous on I ,

(ii) the operator $T: C(I) \rightarrow C(I)$ is a bounded linear operator

(iii) the function $f = f\big(t, x(t)\big) : I \times R \rightarrow R$ is continuous such that $f : I \times R_+ \rightarrow R_+$ and there is a function $m: R_+ \rightarrow R_+$ such that $|f(t, x(t))| \leq m(|x|)$ for $t \in I$,

(iv) $k = k(t,s): I \times I \rightarrow R_+$ is continuous with respect to its both variables t and s such that $k(t,s) < c, \forall\, t, s \in I$, where c is positive constant $c > 0$ and the linear integral operator K generated by $k(t,s)$ maps R_+ into itself,

(v) $d(Tx) \leq \|T\|\, d(x)$ for any nonnegative function $x \in C(I)$,

(vi) The inequality

$$\|g\| + M\|T\| rcm(r) < r$$

has a positive solution r_0 such that $M\|T\| c\, m(r_0) < 1$

Now we can formulate the main existence theorem

Theorem 4. If the assumptions (i) – (vi) are satisfied, then equation (1) has at least one solution $x \in C(I)$.

Proof:

Using our assumptions (i) – (iv) and lemma 3, we can deduce that H is continuous.

Also, let $x \in C(I), |t_2 - t_1| < \delta$, such that $\delta > 0$ and $t_2, t_1 \in I, t_2 > t_1$,we have :

$$|(Hx)(t_2) - (Hx)(t_1)| = |g(t_2) - g(t_1) + \quad + Tx(t_2)\int_0^{t_2} k(t_2,s)f(s,x(s))ds - Tx(t_1)\int_0^{t_1} k(t_1,s)f(s,x(s))ds\Big|$$

$$\le |g(t_2) - g(t_1)| + \Big|Tx(t_2)\int_0^{t_2} k(t_2,s)f(s,x(s))ds -$$
$$- Tx(t_2)\int_0^{t_2} k(t_1,s)f(s,x(s))ds + Tx(t_2)\int_0^{t_2} k(t_1,s)f(s,x(s))ds - \qquad - Tx(t_2)\int_0^{t_1} k(t_1,s)f(s,x(s))ds +$$
Tx(t2)0t1kt1,sfs,xsds–

$$- Tx(t_1)\int_0^{t_1} k(t_1,s)f(s,x(s))ds\Big|$$

$$\le |g(t_2) - g(t_1)| + |Tx(t_2)|\int_0^{t_2}|k(t_2,s) - k(t_1,s)|\,|f(s,x(s))|\,ds +$$

$$+ |Tx(t_2)|\int_{t_1}^{t_2}|k(t_1,s)|\,|f(s,x(s))|\,ds +$$

$$+ |Tx(t_2) - Tx(t_1)|\int_0^{t_1}|k(t_1,s)|\,|f(s,x(s))|\,ds \quad (6)$$

$$< \varepsilon_1 + \|T\|rM\,\varepsilon_2 m(r) + \|T\|r\delta c m(r) + \|T\||x(t_2) - x(t_1)|Mcm(r)$$

Where $|g(t_2) - g(t_1)| < \varepsilon_1, |k(t_2,s) - k(t_1,s)| < \varepsilon_2$

The last estimate yields that the operator H maps $C(I)$ into itself.

Next, For $x \in B_r$ and using the assumption (iii), (vi) and Lemma 3 we have:

$$|Hx(t)| \le |g(t)| + |Tx(t)|\int_0^t|k(t,s)||f(s,x(s))|ds$$

$$\le \|g\| + \|T\|\|x\|\int_0^t|k(t,s)||f(s,x(s))|ds$$

$$\le \|g\| + M\|T\|rcm(r) < r$$

Hence, there is a positive number r_0 with $M\|T\|c\, m(r_0) < 1$ such that the operator H transforms the ball B_{r_0} into itself.

Let

$$B_{r_0}^+ = \{ x \in B_{r_0} : x(t) \ge 0, t \in I \}$$

Note that, $B_{r_0}^+$ is nonempty, bounded, closed and convex subset of B_{r_0}(see [11]).

Furthermore, for a nonempty subset $X \subset B_{r_0}^+$, take a function $x \in X$ and for $\delta > 0$, let $|t_2 - t_1| < \delta$, $t_1, t_2 \in I, t_2 > t_1$, then we have (using inequality (6) and our assumptions):

$$|(Hx)(t_2) - (Hx)(t_1)| \le \omega(g,\varepsilon) + \|T\|r_0\varepsilon Mm(r_0) +$$

$$+\|T\|r_0 c\delta m(r_0) + |Tx(t_2) - Tx(t_1)|Mcm(r_0)$$

$< \omega(g,\varepsilon) + \|T\| r_0 \varepsilon M m(r_0) + \|T\| r_0 c\delta m(r_0) + \|T\|\omega(x,\varepsilon)Mc\, m(r_0).$

As $\varepsilon \to 0$, since g is continuous, we obtain

$$\omega(HX) \le \|T\| Mcm(r_0)\omega(X) \qquad (7)$$

Finally, choose $x \in X$ and $t_2, t_1 \in I$, $t_2 > t_1$ such that $(t_2 - t_1) < \delta$ then we have :

$$|(Hx)(t_2) - (Hx)(t_1)| - [(Hx)(t_2) - (Hx)(t_1)] =$$

$$= \left| g(t_2) - g(t_1) + Tx(t_2)\int_0^{t_2} k(t_2,s) f(s,x(s))ds - \right.$$

$$\left. - Tx(t_1)\int_0^{t_1} k(t_1,s) f(s,x(s))ds \right| - [g(t_2) - g(t_1) +$$

$$+ Tx(t_2)\int_0^{t_2} k(t_2,s) f(s,x(s))ds - Tx(t_1)\int_0^{t_1} k(t_1,s) f(s,x(s))ds]$$

$$\le |g(t_2) - g(t_1)| - [g(t_2) - g(t_1)] +$$

$$+ \left| Tx(t_2)\int_0^{t_2} k(t_2,s) f(s,x(s))ds - Tx(t_1)\int_0^{t_1} k(t_1,s) f(s,x(s))ds \right| -$$

$$- [Tx(t_2)\int_0^{t_2} k(t_2,s) f(s,x(s))ds - Tx(t_1)\int_0^{t_1} k(t_1,s) f(s,x(s))ds] \le |g(t_2) - g(t_1)| - [g(t_2) - g(t_1)] +$$

$$+ |Tx(t_2)| \int_0^{t_2} |k(t_2,s) - k(t_1,s)| \, |f(s,x(s))| ds +$$

$$+ |Tx(t_2)| \int_{t_1}^{t_2} |k(t_1,s)| |f(s,x(s))| ds +$$

$$+ |Tx(t_2) - Tx(t_1)| \int_0^{t_1} |k(t_1,s)| |f(s,x(s))| ds -$$

$$[-Tx(t_2)] \int_0^{t_2} [k(t_2,s) - k(t_1,s)]\, f(s,x(s))ds -$$

$$- [Tx(t_2)] \int_{t_1}^{t_2} k(t_1,s) f(s,x(s))ds -$$

$$- [Tx(t_2) - Tx(t_1)] \int_0^{t_1} k(t_1,s) f(s,x(s))ds)$$

$$\le \{|Tx(t_2) - Tx(t_1)| - [Tx(t_2) - Tx(t_1)\} \left[\int_0^{t_1} k(t_1,s)\, f(s,x(s))ds\right] +$$

$$+ |Tx(t_2)| \int_0^{t_2} |k(t_2,s) - k(t_1,s)| \, |f(s,x(s))| ds +$$

$$+ |Tx(t_2)| \int_{t_1}^{t_2} |k(t_1,s)| |f(s,x(s))| ds +$$

$$+ |Tx(t_2)| \int_0^{t_2} |k(t_2,s) - k(t_1,s)| \, |f(s,x(s))| ds +$$

$$+ |Tx(t_2)| \int_{t_1}^{t_2} |k(t_1,s)| |f(s,x(s))| ds$$

$$\le \{|Tx(t_2) - Tx(t_1)| - [Tx(t_2) - Tx(t_1)]\} [\int_0^{t_1} k(t_1,s) f(s,x(s))ds] +$$

$$+ 2\|T\| r_0 \varepsilon M m(r_0) + 2\|T\| r_0 c\delta m(r_0)$$

The last estimate gives

$$d(\,Hx\,) \le d(Tx)\int_0^{t_1} k(t_1,s)f\big(s,x(s)\big)ds$$

$$\le\ d(Tx)\ \int_0^{t_1}|k(t_1,s)|\ \big|f\big(s,x(s)\big)\big|\ ds$$

$$\le\ cm(r_0)M\|T\|\ d(x)$$

Hence

$$d(\,HX\,) \le \|T\|Mcm(r_0)\ d(X) \qquad (8)$$

Combine (7) and (8) we get

$$\mu(\,HX) \le\ \|T\|Mcm(r_0)\mu(X)$$

Using (vi), then we can apply Darbo fixed point theorem to complete the proof ■

In following we will investigate an example of equation (1)

Example 5. Consider the quadratic integral equation

$$x(t) = g(t) +\ x(t)\int_0^t k(t,s)f(s,x(s))\,,\ \ t \in I = [0,M]$$

In this example, comparing with equation (1), we get

$(Tx)(t) = x(t)$ implies that $T = \mathrm{I}$

For $\delta > 0$ assume that $\|x-y\|\ <\ \delta$ and $x,y\ \in\ C(I)$ we have

$|(Tx)(t) -\ (Ty)(t)| =\ |\mathrm{I}x(t) - \mathrm{I}\ y(t)| = |x(t) -\ y(t)|\ <\ \delta$

This proves that the operator T is continuous.

Since

$$\|\mathrm{I}\| =\ \sup_{x\neq 0}\frac{\|Ix\|}{\|x\|} = \frac{\|x\|}{\|x\|} = 1$$

So, T is bounded

For $X \subset C(I), x \in X$ and $|t_2 -\ t_1| <\ \delta, t_2, t_1 \in [0,M]$, such that $t_2 >\ t_1$ then we have:

$|(Tx)(t_2) -\ (Tx)(t_1)| = |x(t_2) -\ x(t_1)| =\ \omega(Tx,\varepsilon)$

$$\omega(Tx,\varepsilon) = \omega(x,\varepsilon)$$

$$\omega(TX) =\ \omega(X)$$

Also, we have:

$d(Tx,\varepsilon) = |(Tx)(t_2) -\ (Tx)(t_1)| -\ [(Tx)(t_2) -\ (Tx)(t_1)]$

$$= |x(t_2) -\ x(t_1)| -\ [x(t_2) -\ x(t_1)]$$

$= d(x, \varepsilon)$

Hence , the assumptions (ii) and (v) are satisfied. So, under the assumptions (i), (iii), (iv) and (vi) we can apply theorem 4 to get a continuous solutions for our integral equation of example 5 ■

In the sequel, we will investigate the solvability of a fractional integral equation, which in the form

$$x(t) = g(t) + Tx(t) \int_0^t (t-s)^{\alpha-1} k(t,s) f\big(s, x(s)\big) ds, \;\; t \in [0, M] \; (9)$$

Where $0 < \alpha \leq 1$.

Theorem 6. Let the assumptions (i)-(v) of Theorem 4 and the assumption

(vi) inequality $\|g\| + \frac{M^\alpha}{\alpha} \|T\| rcm(r) < r$ has a positive solution r_0 such that $M^\alpha \|T\| c\, m(r_0) < \alpha$,

be satisfied then the integral equation (9) has at least one continuous

solution .

Proof:

Define the operator H associated with the integral equation (9) by

$$(Hx)(t) = g(t) + \; Tx(t) \int_0^t (t-s)^{\alpha-1} k(t,s) f\big(s, x(s)\big) ds, \; t \in [o, M]$$

Using our assumption (i) – (iv), we can deduce that H is continuous.

Let $x \in C(I)$, $|t_2 - t_1| < \delta$ such that $\delta > 0$ and $t_2, t_1 \in I$, $t_2 \geq t_1$, as before we can see that:

$$|Hx(t_2) - Hx(t_1)| \leq |g(t_2) - g(t_1)| +$$

$$+ |Tx(t_2)| \int_0^{t_2} |(t_2 - s)^{\alpha-1}| \, |k(t_2, s) - k(t_1, s)| \big| f\big(s, x(s)\big) \big| ds +$$

$$+ |Tx(t_2)| \int_0^{t_2} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| \, |k(t_1, s)| \big| f\big(s, x(s)\big) \big| ds +$$

$$+ |Tx(t_2)| \int_{t_1}^{t_2} |(t_1 - s)^{\alpha-1}| \, |k(t_1, s)| \big| f\big(s, x(s)\big) \big| ds +$$

$$+ |Tx(t_2) - Tx(t_1)| \int_0^{t_1} |(t_1 - s)^{\alpha-1}| \, |k(t_1, s)| \big| f\big(s, x(s)\big) \big| ds \;\; (10)$$

Using the mean value theorem for the function $(t-s)^{\alpha-1}$, we have

$$|Hx(t_2) - Hx(t_1)| \leq \; \varepsilon_1 + \; \|T\| \|x\| \varepsilon_2 m(r) \left| \frac{{t_2}^\alpha}{\alpha} \right| +$$

$$+ \|T\| \|x\| cm(r) \int_0^{t_2} |(t_2 - \; t_1)(\, \alpha - 1)(z - \; s)^{\alpha-2}| \, ds$$

$$+ \|T\| \|x\| cm(r) \left| \frac{(t_1 - t_2)^\alpha}{\alpha} \right| + \|T\| |x(t_2) - x(t_1)| cm(r) \left| \frac{{t_1}^\alpha}{\alpha} \right|,$$

where $z \epsilon (t_1, t_2)$.

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$|Hx(t_2) - Hx(t_1)| < \varepsilon_1 + \|T\| r \varepsilon_2 m(r) \frac{{t_2}^{\alpha}}{\alpha} +$

$+\|T\| rcm(r) |t_2 - t_1| [\,|z|^{\alpha-1} - |z - t_2|^{\alpha-1}] + \|T\| rcm(r) \left|\frac{(t_1-t_2)^{\alpha}}{\alpha}\right| +$

$$+ \|T\| |x(t_2) - x(t_1)| cm(r) \frac{{t_1}^{\alpha}}{\alpha} = \varepsilon$$

This means that the operator H maps $C(I)$ into itself .

Next, for $x \in B_r$ and using the assumption (iii) and (iv) we have:

$|Hx(t)| \leq \|g\| + \|T\| \|x\| \int_0^t |t-s|^{\alpha-1} |k(t,s)| |f(s, x(s))| \, ds$

$\leq \|g\| + \|T\| r \frac{t^{\alpha}}{\alpha} \, c \, m(r)$

$< \|g\| + \|T\| r \frac{M^{\alpha}}{\alpha} \, c \, m(r)$

Hence, the operator H transforms the ball B_{r_0} into itself such that there is a positive number r_0 with $M^{\alpha} \|T\| c \, m(r_0) < \alpha$.

Let

$B_{r_0}^{+} = \{ x \in B_{r_0} : x(t) \geq 0, t \in I \}$

where $B_{r_0}^{+}$ is nonempty, bounded, closed and convex, as seen before

For a nonempty subset $X \subset B_{r_0}$, take an arbitrary function $x \in X$ and let

$t_2, t_1 \in I, t_2 > t_1$ choose $|t_2 - t_1| < \delta$ then, from (10) we will have:

$$|Hx(t_2) - Hx(t_1)| < |g(t_2) - g(t_1)| + \|T\| r_0 \varepsilon_2 m(r_0) \frac{{t_2}^{\alpha}}{\alpha} +$$

$$+ \|T\| r_0 cm(r_0) |t_2 - t_1| [\,|z|^{\alpha-1} - |z - t_2|^{\alpha-1}] +$$

$$+ \|T\| r_0 cm(r_0) \left|\frac{(t_1-t_2)^{\alpha}}{\alpha}\right| + |Tx(t_2) - Tx(t_1)| cm(r_0) \frac{{t_1}^{\alpha}}{\alpha}$$

$\leq \omega(g, \varepsilon) + \omega(Tx, \varepsilon) cm(r_0) \frac{{t_1}^{\alpha}}{\alpha}$

Since g is continuous then $\omega(g, \varepsilon) \to 0 \; as \; |t_2 - t_1| \to 0$

So, we have

$\omega(Hx, \varepsilon) < \omega(Tx, \varepsilon) cm(r_0) \frac{M^{\alpha}}{\alpha}$

$\leq \|T\| \, \omega(x, \varepsilon) m(r_0) \frac{M^{\alpha}}{c\alpha}$

We obtain

$\omega(HX) \leq cm(r_0) \frac{M^{\alpha}}{\alpha} \|T\| \omega(X)$ (11)

Now, let us take a nonempty set $X \subset B_{r_0}^{+}$ and choose $x \in X$, $t_2, t_1 \in I$, $t_1 \leq t_2$ such that $|t_2 - t_1| < \delta$ then we have as before:

$$|(Hx)(t_2) - (Hx)(t_1)| - [(Hx)(t_2) - (Hx)(t_1)] \leq$$

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$\leq \{|Tx(t_2) - Tx(t_1)| - [Tx(t_2) - Tx(t_1)]\}.$

$. [\int_0^{t_1}(t_1 - s)^{\alpha-1} k(t_1, s) f(s, x(s)) ds] +$

$+ |Tx(t_2)| \int_0^{t_2}|(t_2 - s)^{\alpha-1}| |k(t_2, s) - k(t_1, s)||f(s, x(s))|ds +$

$+ |Tx(t_2)| \int_0^{t_2}|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| |k(t_1, s)||f(s, x(s))|ds +$

$+ |Tx(t_2)| \int_{t_1}^{t_2}|(t_1 - s)^{\alpha-1}| |k(t_1, s)||f(s, x(s))|ds +$

$+|Tx(t_2)| \int_0^{t_2}|(t_2 - s)^{\alpha-1}| |k(t_2, s) - k(t_1, s)||f(s, x(s))|ds +$

$+ |Tx(t_2)| \int_0^{t_2}|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| |k(t_1, s)||f(s, x(s))|ds +$

$+ |Tx(t_2)| \int_{t_1}^{t_2}|(t_1 - s)^{\alpha-1}| |k(t_1, s)||f(s, x(s))|ds$

$\leq \{|Tx(t_2) - Tx(t_1)| - [Tx(t_2) - Tx(t_1)]\}$

$. \left[\int_0^{t_1}(t_1 - s)^{\alpha-1} k(t_1, s) f(s, x(s)) ds\right] + 2\|T\| r_0 \varepsilon_2 m(r_0) \frac{{t_2}^{\alpha}}{\alpha} +$

$+ 2\|T\| r_0 c m(r_0) |t_2 - t_1| [\, |z|^{\alpha-1} - |z - t_2|^{\alpha-1}] +$

$+ 2\|T\| r_0 c m(r_0) \left|\frac{(t_1 - t_2)^{\alpha}}{\alpha}\right|$

$\leq d(Tx) \left[\int_0^{t_1}(t_1 - s)^{\alpha-1} k(t_1, s) f(s, x(s)) ds\right]$

$\leq d(Tx) \left|\int_0^{t_1}(t_1 - s)^{\alpha-1} k(t_1, s) f(s, x(s)) ds\right|$

$\leq d(Tx) \int_0^{t_1}|(t_1 - s)^{\alpha-1}||k(t_1, s)| \, |f(s, x(s))| \, ds$

So, we have

$d(Hx) \leq d(Tx \left|\frac{{t_1}^{\alpha}}{\alpha} c \, m(r_0)\right|)$

$$d(HX) < c \, m(r_0) \frac{M^{\alpha}}{\alpha} \|T\| \, d(X) \quad (12)$$

Combine (11) &(12) we obtain

$$\mu(Hx) \leq c \, m(r_0) \frac{M^{\alpha}}{\alpha} \|T\| \, \mu(x)$$

Applying Darbo fixed point theorem and using (vi) which proves that the equation (9) has at least one solution belonging to the space $C(I)$ ■

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