Certain Classes of Analytic Functions Involving Multiplier Transformations

J. J. Bhamare
Department of Mathematics,
S. S. V. P. S. B. S. Deore, College of Engineering,
Deopur,Dhule-424002
M. S., India.

S. M. Khairnar
Department of Engineering Sciences,
MIT Academy of Engineering, Alandi, Pune-412105,
M. S., India

Abstract- In this paper, we study certain subclasses of analytic functions involving multiplier transformations in the open unit disc. We derive few subordination results for the functions in these subclasses and discuss the applications of subordination results with the help of complex functions. We obtain coefficient estimates, radii of starlikeness, convexity and close-to-convexity, extreme points, and integral means inequalities, growth and distortion theorems for these classes.

Keywords: *multiplier transformations, radii of convexity and close-to-convexity, extreme points, subordination etc.*

AMS Subject Classification: 30C45 & 30C50

*****

I. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{N}$ denote the class of functions of the form

$$f(z)^{\beta} = z^{\beta} + \sum_{n=2}^{\infty} \beta a_n z^{\beta+n-1} \tag{1.1}$$

which are analytic and univalent in the open disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Also consider the subclass $\mathcal{M}$ of $\mathcal{N}$ consisting of functions of the form

$$f(z)^{\beta} = z^{\beta} - \sum_{n=2}^{\infty} \beta a_n z^{\beta+n-1} \quad (\beta > 0\,, \quad a_n \geq 0) \tag{1.2}$$

For any integer m, we defined the multiplier transformations I_m^l of functions $f \in \mathcal{N}$ by

$$\begin{aligned} I_m^l f(z)^{\beta} &= z^{\beta} - \sum_{n=2}^{\infty} \beta \left(\frac{\beta + l}{\beta + l + n - 1}\right)^m a_n z^{\beta+n-1} \\ &= z^{\beta} - \sum_{n=2}^{\infty} \beta Q(n,\beta,l) a_n z^{\beta+n-1}, \quad (l \geq 0, z \in U) \end{aligned} \tag{1.3}$$

$$where\ Q(n,\beta,l)a_n = \left(\frac{\beta+l}{\beta+l+n-1}\right)^m$$

A function $f(z) \in \mathcal{M}$ is said to be in the class $UST(\alpha,k)$ (k-Uniformly starlike functions of order α) if it satisfies the condition:

$$Re\left\{z\frac{f'(z)^{\beta}}{f(z)^{\beta}} - \alpha\right\} > k\left|z\frac{f'(z)^{\beta}}{f(z)^{\beta}} - 1\right|, 0 \leq \alpha < 1, k \geq 0\ and\ z \in U, \tag{1.4}$$

And is said to be in the class $UCV(\alpha,k)$ (k-Uniformly convex functions of order α) if it satisfies the condition:

$$Re\left\{1 + z\frac{f''(z)^{\beta}}{f'(z)^{\beta}} - \alpha\right\} > k\left|z\frac{f''(z)^{\beta}}{f'(z)^{\beta}} - 1\right|, 0 \leq \alpha < 1, k \geq 0\ and\ z \in U, \tag{1.5}$$

Indeed it follows from (4) and (5) that

$$f \in\ UCV(\alpha,k) \Leftrightarrow zf' \in UST(\alpha,k) \tag{1.6}$$

The interesting geometric properties of these function classes were extensively studied by Waggas Galib Astan et. Al [16], Kanas et al. in [14, 15], and Murugusundaramoorthy and Magesh [3,4], Astan and Kulkarni[17].Astan and Buti[18], S. M. Khairnar and N. H. More [12], S. M. Khairnar and Meena More [10, 11].

Definition1.1: For two functions f and g analytic in U, we say that the function f is subordinate to g in U, denoted by $f \prec g$, if their exist a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < |z| < 1$ $(z \in U)$, such that $f(z) = g\big(w(z)\big)$ $(z \in U)$.
In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0), f(U) = g(U)$.
The Littlewood's subordination theorem which we will use in our investigation to obtain the integral mean inequality.
Lemma1.2: If the functions $f(z)$ and $g(z)$ are analytic in U, with $f(z) \prec g(z)$, then

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|g(re^{i\theta})\right|^{\eta} d\theta \tag{1.7}$$

Where $\eta > 0,\ z = re^{i\theta}\ and\ 0 < r < 1$. Strict inequality holds for $0 < r < 1$
Unless f is constant or $w(z) = \alpha z, |\alpha| = 1$.
Let $\varphi(z)$ be an analytic function in U with $\varphi(0) = 1,\ \varphi'(0) > 0\ and\ R\big(\varphi(z)\big) > 0, z \in U.$
which maps the open unit disc U onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Then, by$S^*(\varphi)\ and\ C(\varphi)$, respectively, denote the subclass of normalized analytic function class A which satisfy the following subordination relation

$$z\frac{f'(z)^\beta}{f(z)^\beta} \prec \varphi(z)\ and\ 1 +\ z\frac{f''(z)^\beta}{f'(z)^\beta} \prec \varphi(z), z \in U.$$

These functions introduced and studied by Ma and Minda. For

$$\varphi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}\ \ (z \in U, 0 \leq \alpha < 1);$$

These function classes reduce to the well known classes $S^*(\alpha)$ of starlike functions of order α in U and $C(\alpha)$ of convex functions of order α in U.

Definition1: For $\beta > 0, -1 \leq \alpha < 1, l, k \geq 0\ and\ I_m^l f(z)^\beta\ given\ by\ (1.3)$, we difined a new class $\mathcal{N}_m^l(\alpha, \beta, \lambda, k)$ as subclass of consisting function $f(z)^\beta$ of the form (1.2) and satisfying analytic criterion

$$Re\left\{\frac{z\left(I_m^l f(z)^\beta\right)'}{\lambda z(I_m^l f(z)^\beta)' + (1 - \lambda)(I_m^l f(z)^\beta)} - \alpha\right\} \geq k\left|\frac{z\left(I_m^l f(z)^\beta\right)'}{\lambda z(I_m^l f(z)^\beta)' + (1 - \lambda)(I_m^l f(z)^\beta)} - 1\right|. \tag{1.8}$$

II. COEFFICIENT ESTIMATE

We obtain Necessary and Sufficient condition for the function $f(z)^\beta$ of the form (1.2) to belong to the class $\mathcal{N}_m^l(\alpha, \beta, \lambda, k)$

Theorem2.1: A function $f(z)^\beta$ is in the class $\mathcal{N}_m^l(\alpha, \beta, \lambda, k)$if and only if

$$\sum_{n=1}^{\infty}\{[(1 + k) - \lambda(k + \alpha)](\beta + n - 1) - (k + \alpha)(1 - \lambda)\}\beta Q(n, \beta, l)a_n \leq (1 + k)\beta - (k + \alpha)\big(\lambda\beta + (1 - \lambda)\big) \tag{2.1}$$

The result is sharp for the function

$$a_n = \frac{(1 + k)\beta - (k + \alpha)(\lambda\beta + 1 - \lambda)}{\{[(1 + k) - \lambda(k + \alpha)](\beta + n - 1) - (k + \alpha)(1 - \lambda)\}\beta Q(n, \beta, l)}$$

Where $\beta > 0, -1 \leq \alpha < 1, k, l \geq 0, 0 \leq \lambda < 1$

$$I_m^l f(z)^\beta = z^\beta - \sum_{n=2}^{\infty} \frac{(1 + k)\beta - (k + \alpha)(\lambda\beta + 1 - \lambda)}{\{[1 + k - \lambda(k + \alpha)](\beta + n - 1) - (k + \alpha)(1 - \lambda)\}Q(n, \beta, l)} z^{\beta + n - 1},$$

Proof: Suppose that (2.1) is true for $z \in U$. Then

$$Re\left\{\frac{z\left(I_m^l f(z)^\beta\right)'}{\lambda z(I_m^l f(z)^\beta)' + (1 - \lambda)(I_m^l f(z)^\beta)} - \alpha\right\} \geq k\left|\frac{z\left(I_m^l f(z)^\beta\right)'}{\lambda z(I_m^l f(z)^\beta)' + (1 - \lambda)(I_m^l f(z)^\beta)} - 1\right|$$

$$Re\left\{\frac{[\beta-\alpha(\lambda\beta+1-\lambda)]-\sum_{n=1}^{\infty}\beta Q(\mathrm{n},\beta,\mathrm{l})[(\beta+\mathrm{n}-1)-\alpha\lambda(\beta+\mathrm{n}-1)-\alpha(1-\lambda)]a_n z^{n-1}}{(\lambda\beta+1-\lambda)-\sum_{n=1}^{\infty}\beta Q(\mathrm{n},\beta,\mathrm{l})[\lambda(\beta+\mathrm{n}-1)+(1-\lambda)]a_n z^{n-1}}\right\}$$

$$\geq k\left|\frac{[\beta-(\lambda\beta+1-\lambda)]-\sum_{n=1}^{\infty}\beta Q(\mathrm{n},\beta,\mathrm{l})[(\beta+\mathrm{n}-1)-\lambda(\beta+\mathrm{n}-1)-(1-\lambda)]a_n z^{n-1}}{(\lambda\beta+1-\lambda)-\sum_{n=1}^{\infty}\beta Q(\mathrm{n},\beta,\mathrm{l})[\lambda(\beta+\mathrm{n}-1)+(1-\lambda)]a_n z^{n-1}}\right|$$

Allowing $z \to 1$ along with real axis, we get

$$\sum_{n=1}^{\infty}\{[1+k-(k+\alpha)\lambda](\beta+\mathrm{n}-1)-(k+\alpha)(1-\lambda)\}\beta Q(\mathrm{n},\beta,\mathrm{l})a_n \leq [(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)]$$

Conversely, assume that $f \in \mathcal{N}_m^l(\alpha,\beta,\lambda,k)$, *then*

$$k\left|\frac{z\left(I_m^l f(z)^\beta\right)'}{\lambda z(I_m^l f(z)^\beta)'+(1-\lambda)(I_m^l f(z)^\beta)}-1\right|-Re\left\{\frac{z\left(I_m^l f(z)^\beta\right)'}{\lambda z(I_m^l f(z)^\beta)'+(1-\lambda)(I_m^l f(z)^\beta)}-\alpha\right\}$$

$$\leq (1+k)\left|\frac{z\left(I_m^l f(z)^\beta\right)'}{\lambda z(I_m^l f(z)^\beta)'+(1-\lambda)(I_m^l f(z)^\beta)}-1\right|$$

$$\leq (1+k)\left|\frac{\left(\beta-\lambda\beta-(1-\lambda)\right)-\sum_{n=1}^{\infty}\beta Q(\mathrm{n},\beta,\mathrm{l})[(\beta+\mathrm{n}-1)(1-\lambda)-(1-\lambda)]a_n z^{n-1}}{\left(\lambda\beta+(1-\lambda)\right)-\sum_{n=1}^{\infty}\beta Q(\mathrm{n},\beta,\mathrm{l})[(\beta+\mathrm{n}-1)\lambda+(1-\lambda)]a_n z^{n-1}}\right|$$

is bounded above if $\leq (1-\alpha)$, *Letting* $z \to 1^-$ *along the real axis, we have*

$$\leq (1+k)\left|\frac{\left(\beta-\lambda\beta-(1-\lambda)\right)-\sum_{n=1}^{\infty}\beta Q(\mathrm{n},\beta,\mathrm{l})[(\beta+\mathrm{n}-1)(1-\lambda)-(1-\lambda)]a_n}{\left(\lambda\beta+(1-\lambda)\right)-\sum_{n=1}^{\infty}\beta Q(\mathrm{n},\beta,\mathrm{l})[(\beta+\mathrm{n}-1)\lambda+(1-\lambda)]a_n}\right|$$

$$\sum_{n=1}^{\infty}\{(\beta+\mathrm{n}-1)[(1+k)-\lambda(k+\alpha)]-(k+\alpha)(1-\lambda)\}\beta Q(\mathrm{n},\beta,\mathrm{l})a_n \leq (1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)$$

which completes the proof.

III. RADII OF STARLIKENESS, CONVEXITY AND CLOSE-TO-CONVEXITY

We obtain the radii of convexity and close-to- convexity results for $f(z)$ functions in the class$\mathcal{N}_m^l(\alpha,\beta,\lambda,k)$ in the following theorem

Theorem3.1: Let $f \in \mathcal{N}_m^l(\alpha,\beta,\lambda,k)$. Then f is starlike of order δ, $(0 \leq \delta < 1)$ in the disk $|z| < r = r_1(\alpha,\beta,\lambda,k,n,\delta)$, where

$$r_1 = \inf_{n \geq 2}\left[\frac{(\beta-\delta)\{[(1+k)-\lambda(k+\alpha)](\beta+\mathrm{n}-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)}{(\beta+n-\delta-1)[(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)]}\right]^{1/n-1} \tag{3.1}$$

Proof: For $0 \leq \delta < 1$, It is sufficient to show that

$$\left|\frac{f'(z)^\beta}{f(z)^\beta}-1\right| \leq 1-\delta\,, \tag{3.2}$$

For $|z| \leq r_1$, we have

$$\left|\frac{Zf'(z)^\beta}{f(z)^\beta}-1\right| \leq \left|\frac{(\beta-1)-\sum_{n=2}^{\infty}\beta(\beta+n-2)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\beta a_n z^{n-1}}\right| \leq 1-\delta \tag{3.3}$$

Hence, (3.3) holds true if

$$\sum_{n=2}^{\infty}\frac{(\beta+n-\delta-1)\beta}{(\beta-\delta)}|a_n||z|^{n-1} \leq 1 \tag{3.4}$$

With the aid of (2.1) and (3.4) is true if

$$|z|^{n-1} \le \frac{(\beta-\delta)\{[(1+k)-\lambda(k+\alpha)](\beta+n-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)}{(\beta+n-\delta-1)[(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)]} \tag{3.5}$$

Therefore,

$$|z| \le \left[\frac{(\beta-\delta)\{[(1+k)-\lambda(k+\alpha)](\beta+n-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)}{(\beta+n-\delta-1)[(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)]}\right]^{1/n-1} \tag{3.6}$$

Setting $|z| = r_1(\alpha,\beta,\lambda,k,n,\delta)$in (3.6), we get the radius of starlikeness, which completes the proof of theorem 3.1.

Theorem3.2: Let $f \in \mathcal{N}_m^l(\alpha,\beta,\lambda,k)$. Then f is convex of order $\delta, (0 \le \delta < 1)$ in the disk $|z| < r = r_2(\alpha,\beta,\lambda,k,n,\delta)$, where

$$r_2 = \inf_{n \ge 2}\left[\frac{(\beta-\delta)\{[(1+k)-\lambda(k+\alpha)](\beta+n-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)}{(\beta+n-1)(\beta+n-\delta-1)[(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)]}\right]^{1/n-1} \tag{3.7}$$

Proof: For $0 \le \delta < 1$, it is sufficient to show

$$\left|z\frac{f''(z)^\beta}{f'(z)^\beta}\right| \le 1-\delta\,, \tag{3.8}$$

For $|z| \le r_2$, we have

$$\left|z\frac{f''(z)^\beta}{f'(z)^\beta}\right| \le \left|\frac{(\beta-1)-\sum_{n=2}^{\infty}(\beta+n-1)(\beta+n-2)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}(\beta+n-1)a_n z^{n-1}}\right| \le 1-\delta\,, \tag{3.9}$$

Hence, (3.9) holds true if

$$\sum_{n=2}^{\infty}\frac{(\beta+n-1)(\beta+n-\delta-1)}{(\beta-\delta)}|a_n||z|^{n-1} \le 1 \tag{3.10}$$

With the aid of (2.1) and (3.4) is true if

$$|z|^{n-1} \le \frac{(\beta-\delta)\{[(1+k)-\lambda(k+\alpha)](\beta+n-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)}{(\beta+n-1)(\beta+n-\delta-1)[(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)]} \tag{3.11}$$

$$|z| \le \left[\frac{(\beta-\delta)\{[(1+k)-\lambda(k+\alpha)](\beta+n-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)}{(\beta+n-1)(\beta+n-\delta-1)[(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)]}\right]^{1/n-1} \tag{3.12}$$

Setting $z = r_2(\alpha,\beta,\lambda,k,n,\delta)$in (3.14), we get the radius of convexity, which completes the proof of theorem 3.2.

Theorem3.3: Let $f \in \mathcal{N}_m^l(\alpha,\beta,\lambda,k)$. Then f is close to convex of order $\delta, (0 \le \delta < 1)$ in the disk $|z| < r = r_3(\alpha,\beta,\gamma,n,\delta)$, where

$$r_3 = \inf_{n \ge 2}\left[\frac{(\beta-\delta)\{[(1+k)-\lambda(k+\alpha)](\beta+n-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)}{(\beta+n-\delta-1)[(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)]}\right]^{1/n-1} \tag{3.13}$$

Proof: For $0 \le \delta < 1$, it is sufficient to show that

$$\left|\frac{f'(z)^\beta}{z^{\beta-1}}-1\right| \le 1-\delta \tag{3.14}$$

For $|z| \le r_3$, we have

$$\left|\frac{f'(z)^\beta}{z^{\beta-1}}-1\right| \le \left|(\beta-1)+\sum_{n=2}^{\infty}\beta(\beta+n-1)a_n z^{n-1}\right| \le 1-\delta \tag{3.15}$$

ISSN: 2349-7955
008 – 015

Hence, (3.15) holds true if

$$\sum_{n=2}^{\infty} \frac{\beta(\beta+n-1)}{(\beta-\delta)} |a_n||z|^{n-1} \leq 1 \tag{3.16}$$

With the aid of (2.1) and (3.16) is true if

$$|z|^{n-1} \leq \frac{(\beta-\delta)\{[(1+k)-\lambda(k+\alpha)](\beta+\mathrm{n}-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)}{(\beta+n-1)[(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)]} \tag{3.17}$$

Therefore,

$$|z| \leq \left\{\frac{(\beta-\delta)\{[(1+k)-\lambda(k+\alpha)](\beta+\mathrm{n}-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)}{(\beta+n-1)[(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)]}\right\}^{1/n-1} \tag{3.16}$$

Setting $z = r_3(\alpha, \beta, \lambda, k, n, \delta)$in (3.21), we get the radius of close-to-convexity, which completes the proof of theorem 3.3.

IV. EXTREAM POINTS:

The extream points of the class $\mathcal{N}_m^l(\alpha, \beta, \lambda, k)$ are given by the following theorem.

Theorem 4.1: Let $f_1(z)^\beta = z^\beta$,

$$f_n(z)^\beta = z^\beta - \beta \frac{(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)}{\{[(1+k)-\lambda(k+\alpha)](\beta+\mathrm{n}-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)} z^{\beta+n-1} \tag{4.1}$$

For n=2,3,4,…

Then, $f \in \mathcal{N}_m^l(\alpha, \beta, \lambda, k)$if and only if it can be expressed in the form

$$f(z)^\beta = \sum_{n=1}^{\infty} Y_n f_n(z)^\beta \tag{4.2}$$

Where

$$Y_n \geq 0 \; and \sum_{n=1}^{\infty} Y_n = \tag{4.3}$$

Proof: Suppose that f can be express as in (4.2). Our aim is to show that $f \in \mathcal{N}_m^l(\alpha, \beta, \lambda, k)$.
By (4.2), we have that

$$f(z)^\beta = \sum_{n=1}^{\infty} Y_n f_n(z)^\beta = Y_1 f_1(z)^\beta + \sum_{n=2}^{\infty} Y_n f_n(z)^\beta$$

$$= Y_1 f_1(z)^\beta$$
$$+ \sum_{n=2}^{\infty} Y_n \left(z^\beta - \beta \frac{(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)}{\{[(1+k)-\lambda(k+\alpha)](\beta+\mathrm{n}-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)} z^{\beta+n-1}\right) \tag{4.4}$$

$$= \sum_{n=1}^{\infty} Y_n z^\beta - Y_n \beta \frac{(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)}{\{[(1+k)-\lambda(k+\alpha)](\beta+\mathrm{n}-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)} z^{\beta+n-1}$$

$$= z^\beta - \sum_{n=2}^{\infty} \beta Y_n \frac{(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)}{\{[(1+k)-\lambda(k+\alpha)](\beta+\mathrm{n}-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)} z^{\beta+n-1}.$$

Now,

ISSN: 2349-7955
008 – 015

$$\sum_{n=2}^{\infty}\frac{\{[(1+k)-\lambda(k+\alpha)](\beta+n-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)}{(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)} \times Y_n\frac{(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)}{\{[(1+k)-\lambda(k+\alpha)](\beta+n-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)}z^{\beta+n-1}$$

$$=\sum_{n=2}^{\infty}Y_n=1-Y_1\leq 1 \tag{4.5}$$

Thus, $f\in\mathcal{N}_m^l(\alpha,\beta,\lambda,k)$.

Conversely, assume that $f\in\mathcal{N}_m^l(\alpha,\beta,\lambda,k)$. Since

$$a_n\leq\frac{(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)}{\{[(1+k)-\lambda(k+\alpha)](\beta+n-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)}\quad (n\geq 2) \tag{4.6}$$

We can set

$$Y_n=\frac{\{[(1+k)-\lambda(k+\alpha)](\beta+n-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)}{(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)}a_n\ (n\geq 2) \tag{4.7}$$

$$Y_1=1-\sum_{n=2}^{\infty}Y_n$$

Then,

$$f(z)^\beta=z^\beta-\sum_{n=2}^{\infty}\beta\, a_n z^{\beta+n-1}$$

$$=z^\beta-\sum_{n=2}^{\infty}\beta\, Y_n\frac{(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)}{\{[(1+k)-\lambda(k+\alpha)](\beta+n-1)-(k+\alpha)(1-\lambda)\}Q(n,\beta,l)}z^{\beta+n-1}$$

$$=z^\beta-\sum_{n=2}^{\infty}Y_n\left(z^\beta-f_n(z)^\beta\right) \tag{4.8}$$

$$=Y_1f_1(z)^\beta+\sum_{n=2}^{\infty}Y_nf_n(z)^\beta=\sum_{n=1}^{\infty}Y_nf_n(z)^\beta$$

This completes the proof of Theorem4.1

V. INTEGRAL MEANS:

Theorem 5.1: Let $\eta>0$. If $f\in\mathcal{N}_m^l(\alpha,\beta,\lambda,k)$ *and* $\{\varphi(\alpha,\beta,n)\}_{n=1}^{\infty}$ are nondecreasing sequences, then, for $z=re^{i\theta}$ *and* $0<r<1$

$$\int_0^{2\pi}\left|f(re^{i\theta})^\beta\right|^\eta d\theta\leq\int_0^{2\pi}\left|f_2(re^{i\theta})^\beta\right|^\eta d\theta \tag{5.1}$$

where

$$f_2(z)^\beta = z^\beta - \beta\frac{\big[(1+k)\beta-(k+\alpha)\big(\lambda\beta+(1-\lambda)\big)\big]}{\varphi(\alpha,\beta,\lambda,k,2)}z^{\beta+1} \tag{5.2}$$

$$\varphi(\alpha,\beta,\lambda,k,n) = \big[[(1+k)-\lambda(k+\alpha)](\beta+n-1)-(k+\alpha)(1-\lambda)\big]Q(n,\beta,l).$$

Proof: Let f of the form of (1.2) and

$$f_2(z)^\beta = z^\beta - \beta\frac{\big[(1+k)\beta-(k+\alpha)\big(\lambda\beta+(1-\lambda)\big)\big]}{\varphi(\alpha,\beta,\lambda,k,2)}z^{\beta+1} \tag{5.3}$$

Then we must show that

$$\int_0^{2\pi}\left|1-\sum_{n=1}^{\infty}\beta a_n z^{n-1}\right|^\eta d\theta \le \int_0^{2\pi}\left|1-\beta\frac{\big[(1+k)\beta-(k+\alpha)\big(\lambda\beta+(1-\lambda)\big)\big]}{\varphi(\alpha,\beta,\lambda,k,2)}z\right|^\eta d\theta \tag{5.4}$$

By lemma 5.2, it suffices to show that

$$1-\sum_{n=1}^{\infty}\beta a_n z^{n-1} \prec 1-\beta\frac{\big[(1+k)\beta-(k+\alpha)\big(\lambda\beta+(1-\lambda)\big)\big]}{\varphi(\alpha,\beta,\lambda,k,2)}z \tag{5.5}$$

Setting

$$1-\sum_{n=1}^{\infty}\beta a_n z^{n-1} = 1-\beta\frac{\big[(1+k)\beta-(k+\alpha)\big(\lambda\beta+(1-\lambda)\big)\big]}{\varphi(\alpha,\beta,\lambda,k,2)}w(z) \tag{5. 6}$$

From (5.7) and (2.1) we obtain

$$\begin{aligned}|w(z)\,| &= \left|\sum_{n=2}^{\infty}\frac{\varphi(\alpha,\beta,\lambda,k,2)}{(\beta+1)(1-\gamma)+(1-\alpha)}a_n z^{n-1}\right| \\ &\le |z|\sum_{n=2}^{\infty}\frac{\varphi(\alpha,\beta,\lambda,k,2)}{(1+k)\beta-(k+\alpha)\big(\lambda\beta+(1-\lambda)\big)}a_n\end{aligned} \tag{5.7}$$

$$\le |z| < 1.$$

This completes the proof of Theorem 5.1.

VI. GROWTH AND DISTORSION THEOREM:

Theorem 6.1: If $f \in \mathcal{N}_m^l(\alpha,\beta,\lambda,k)$, then,

$$|z|^\beta - \frac{\beta[(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)]}{\varphi(\alpha,\beta,\lambda,k,2)}|z|^{\beta+1} \le f(z)^\beta \le |z|^\beta + \frac{\beta[(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)]}{\varphi(\alpha,\beta,\lambda,k,2)}, \tag{6.1}$$

Where $\varphi(\alpha,\beta,\lambda,k,n) = \big[[(1+k)-\lambda(k+\alpha)](\beta+n-1)-(k+\alpha)(1-\lambda)\big]Q(n,\beta,l)$

This results are sharp

Theorem 6.1: If $f \in \mathcal{N}_m^l(\alpha,\beta,\lambda,k)$, then,

$$-\frac{\beta[(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)](\beta+1)}{\beta|z|^{\beta-1}\varphi(\alpha,\beta,\lambda,k,2)}|z|^{\beta} \leq f^{'(z)^{\beta}} \leq \beta|z|^{\beta-1}+\frac{\beta[(1+k)\beta-(k+\alpha)(\lambda\beta+1-\lambda)](\beta+1)}{\varphi(\alpha,\beta,\lambda,k,2)}|z|^{\beta}, \tag{6.2}$$

Where $\varphi(\alpha,\beta,\lambda,k,n)=\big[[(1+k)-\lambda(k+\alpha)](\beta+\mathrm{n}-1)-(k+\alpha)(1-\lambda)\big]Q(n,\beta,l)$

This results are sharp

References:

[1] A. W. Goodman, On uniformly convex functions, Ann. Polon. Math. 56 (1991), 87–92.

[2] G. Murugusundaramoorthy and N. Magesh, An application of second order differential inequalities based on linear and integral operators, International J. of Math. Sci.and Engg. Appls. (IJMSEA) 2 (2008), 105–114.

[3] G. Murugusundaramoorthy and N. Magesh, Starlike and convex functions of complex order involving the Dziok-Srivastava operator, Integral Transforms and Special Functions, vol. 18, no.(2007), 5-6, pp. 419–425.

[4] G. Murugusundaramoorthy and N. Magesh, Certain subclasses of starlike functions of complex order involving generalized hypergeometric functions, International Journal of Mathematics and Mathematical Sciences, vol. 2010, Article ID 178605, 12 pages.

[5] H. Silverman, Integral means for univalent functions with negative coefficients, Houston Journal of Mathematics, vol. 23,(1997), no. 1, pp. 169–174.

[6] J. E. Littlewood, On inequalities in theory of functions, Proceedings of the London Mathematical Society, vol. 23,(1925), pp. 481–519.

[7] N. E. Cho and T. H. Kim, Multiplier transformations and strongly close-to-convex functions, Bulletin of the Korean Mathematical Society, vol. 40,(2003), no. 3, pp. 399–410.

[8] P. L. Duren, Univalent functions, Grundlehren der Mathematischen Wissenchaften,

[9] Vol. 259, Springer-Verlag, New York, 1983.

[10] R. M. El-Ashwah and M. K. Aouf, New classes of p-valent harmonic functions, Bulletin of Mathematical Analysis and Applications, vol. 2(2010), no. 3, pp. 53–64.

[11] S. M. Khairnar and Meena More, A subclass of uniformly convex functions associated with certain fractional calculus operators, IAENG, International Journal of Applied Mathematics, 39 (2009), IJAM-39-07.

[12] S. M. Khairnar and Meena More, Properties of a class of analytic and univalent functions using Ruscheweyh derivative, Int. Journal of Math. Analysis 3 (2008), 967–976.

[13] S. M. Khairnar and N. H. More, A subclass of uniformly convex functions and a corresponding subclass of starlike functions with fixed second coefficient defined by Carlson and Shaffer operator, International Mathematical Forum,5,2010, no. 37, 1839-1848

[14] S. M. Khairnar and Jayashri J. Bhamare, On properties of uniformly convex functions defined using Carlson- Shaffer operator, Shekhar(new Series) International Journal of Mathematics, Vol.1,(2009), no.1, pp.67-84 .

[15] S. Kanas and A. Wisniowska, Conic regions and k-uniform convexity, Journal of Computational and Applied Mathematics, vol. 105,(1999), no. 1-2, pp. 327–336.

[16] S. Kanas and T. Yaguchi, Subclasses of k-uniformly convex and starlike functions defined by generalized derivative. II, Publications Institute Mathematique, vol. 69(2001), no. 83, pp. 91–100.

[17] Waggas Galib Astan and Ali Hamza Abada, On Subclass of k-uniformly convex functions of complex order involving multiplier transformations, Hindawi Publishing Corporation, Abstract and Applied Analysis, Vol.2012, Article ID 150571,10 pages.

[18] W. G. Atshan and S. R. Kulkarni, Neighborhoods and partial sums of subclass of k-unifor- mly convex functions and related class of k-starlike functions with negative coefficients based on integral operator, Southeast Asian Bulletin of Mathematics, vol. 33, (2009), no. 4, pp. 623–637.

[19] W. G. Atshan and R. H. Buti, On generalized hypergeometric functions and associated classes of k-uniformly convex and k-starlike p-valent functions, Advances and Applications in Mathematical Sciences, vol. 6, (2010), no. 2, pp. 149–160.